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Erika Fecková Škrabuľáková

Quasi-Flett's mean value theorems

Jana Borzová [1] **and Ondrej Hutník**[2]

*Dedicated to the memory of our teacher Ján Ohriska (*1942–†2018).*

Abstract: The paper deals with various extensions of the mean value theorem of differential and integral calculus due to Thomas M. Flett. We provide an overview of results of Flett's type with weakening the differentiability assumption. Considering Dini's derivative, symmetric derivative and v-derivative we get new mean value theorems which we call *quasi-Flett's* mean value theorems.

Keywords: Mean value; Differentiability; Symmetric derivative; Flett's theorem; v-derivative.

Mathematics Subject Classification: 26A24, 26D20

1 Introduction and preliminaries

In this paper we will use the following unified notation: $\mathcal{C}(M)$, resp. $\mathcal{D}^n(M)$, will mean the classes of continuous, resp. n-times differentiable real functions on a set $M \subseteq \mathbb{R}$. Usually we will work with compact sets M in $\mathbb{R}$. For functions f, g on an interval $\langle a, b\rangle$ (for which the following expression has its sense) the expressions of the form

$$\frac{f^{(n)}(b) - f^{(n)}(a)}{g^{(m)}(b) - g^{(m)}(a)}, \quad m, n \in \mathbb{N} \cup \{0\},$$

will be denoted by the symbol ${}_a^b\mathcal{K}\left(f^{(n)}, g^{(m)}\right)$. We use the usual convention $h^{(0)} := h$, and thus for the function $g(x) = x$ on $\langle a, b\rangle$ and $m = 0$ we will write only ${}_a^b\mathcal{K}\left(f^{(n)}\right)$.

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In 1958 Thomas Muirhead Flett (1923–1976) published a short paper [2] in which he gave a variation on the theme of Rolle's theorem where the condition $f(a) = f(b)$ is replaced by $f'(a) = f'(b)$.

Theorem 1 (Flett, 1958)**.** *If $f \in \mathcal{D}\langle a, b\rangle$ and $f'(a) = f'(b)$, then there exists $\eta \in (a, b)$ such that*

$$f'(\eta) = {}_a^\eta\mathcal{K}(f). \tag{1}$$

Alternatively, Flett's result may be seen as Lagrange's type mean value theorem with Rolle's type condition. Geometrically, if a curve $y = f(x)$ has a tangent at each point of $\langle a, b\rangle$ and tangents at the end points $[a, f(a)]$ and $[b, f(b)]$ are parallel, then Flett's theorem guarantees the existence of such a point $\eta \in (a, b)$ that the tangent constructed to the graph of f at that point passes through the point $[a, f(a)]$.

Further sufficient conditions for validity of (1) are investigated in [4]. We summarize some of them: for $f \in \mathcal{D}\langle a, b\rangle$ the assertion of Flett's theorem holds in each of the following cases:

(i) $f'(a) = f'(b)$ (Flett's condition);

(ii) $(f'(a) - {}_a^b\mathcal{K}(f)) \cdot (f'(b) - {}_a^b\mathcal{K}(f)) \geq 0$ (Trahan's condition);

(iii) $\frac{f(a)+f(b)}{2} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$ (Tong's condition);

(iv) $(f'(a) - {}_a^b\mathcal{K}(f)) \cdot f''(a) > 0$ provided $f''(a)$ exists (Maleševič's condition);

(v) $\frac{f(a)+f(b)}{2} = f(\frac{a+b}{2})$ (Tan-Li's condition).

Continuing in extension of Rolle's condition to higher-order derivatives Pawlikowska [6] obtained the following generalization of Flett's theorem.

Theorem 2 (Pawlikowska, 1999)**.** *If $f \in \mathcal{D}^n\langle a, b\rangle$ with $f^{(n)}(a) = f^{(n)}(b)$, then there exists $\eta \in (a, b)$ such that*

$${}_a^\eta\mathcal{K}(f) = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i!}(\eta - a)^{i-1} f^{(i)}(\eta).$$

Clearly, Theorem 2 reduces to Theorem 1 for $n = 1$. There are also results eliminating/replacing the condition $f^{(n)}(a) = f^{(n)}(b)$, see [4]. All these results may be extended at least in two directions: to move from the real line to more general spaces, and/or to consider other types of differentiability of considered functions. In this paper we deal with the latter case and we give results for three generalizations of differentiability: Dini's derivable numbers, Ohriska's v-derivative and symmetric derivative. Since some needed results are not well known, we provide full proofs for the sake of completeness and for the convenience of the reader.

2 Flett's theorem with generalized derivatives

Let f be defined in a neighbourhood $\mathcal{O}(x_0)$ of a point x_0. The concept of derivative of f at x_0 in the sense of proper limit

$$f'(x_0) := \lim_{x \to x_0} {}_{x_0}^{x}\mathcal{K}(f)$$

is very useful. However, since the derivative may fail to exist, it seems desirable to have expressions which may serve us when there is no derivative at a point. For this reason several replacements of derivative have appeared in different contexts: Dini's derivative, symmetric derivative, Peano derivative, etc. The aim of this section is to provide further extensions of Flett's theorem (that originally requires the existence of the derivative f') for not necessarily differentiable functions. In each case we prove a variant of Flett's theorem commonly called quasi-Flett's theorems.

2.1 Flett's theorem with Dini's derivatives

Probably the most elementary replacement of nonexistence of derivative at a point are Dini's derivatives that exist at each point of a function defined on an open interval.

Definition 3. Let f be a real function defined in a neighbourhood of a point x_0. Four *Dini's derivatives* of f at x_0 are given by

(i) $D^+f(x_0) := \limsup\limits_{x \to x_0^+} {}_{x_0}^{x}\mathcal{K}(f)$ (upper right Dini's derivative);

(ii) $D_+f(x_0) := \liminf\limits_{x \to x_0^+} {}_{x_0}^{x}\mathcal{K}(f)$ (lower right Dini's derivative);

(iii) $D^-f(x_0) := \limsup\limits_{x \to x_0^-} {}_{x_0}^{x}\mathcal{K}(f)$ (upper left Dini's derivative);

(iv) $D_-f(x_0) := \liminf\limits_{x \to x_0^-} {}_{x_0}^{x}\mathcal{K}(f)$ (lower left Dini's derivative).

It is easy to verify that $f'(x_0)$ exists if and only if all four Dini's derivatives of f at x_0 give the same value, and one-sided derivative exists if and only if upper and lower Dini's derivative (from the corresponding side) are equal. For our purposes the following result will be important, we refer [7] for the proof.

Proposition 4. *Let $f : \langle a, b\rangle \to \mathbb{R}$ such that $f(a) = f(b)$ and possess minimum m and maximum M on the interval $\langle a, b\rangle$. If $m < f(a) = f(b)$, then there exists $\eta \in (a, b)$ such that*

$$D^-f(\eta) \le 0 \le D_+f(\eta).$$

If $M > f(a) = f(b)$, then there exists $\xi \in (a, b)$ such that

$$D^+f(\xi) \leq 0 \leq D_-f(\xi).$$

A generalization of Flett's theorem with Dini's derivatives is as follows. Recall that a function f defined on a neighbourhood $\mathcal{O}(x_0)$ of a point x_0 is *strictly increasing at* x_0 if there exists $\delta > 0$ such that for $x \in (x_0 - \delta, x_0) \cap \mathcal{O}(x_0)$ it holds $f(x) < f(x_0)$ and for $x \in (x_0, x_0 + \delta) \cap \mathcal{O}(x_0)$ it holds $f(x) > f(x_0)$.

Theorem 5 (Lakshminarasimhan, 1966)**.** *Let $f \in \mathcal{C}\langle a, b\rangle$ and f be differentiable at points a and b. If all four Dini's derivatives of f on the interval (a, b) are finite and $f'(a) = f'(b)$, then there exist $\eta, \xi \in (a, b)$ such that*

$$D^+f(\eta) \leq {}_a^\eta\mathcal{K}(f) \leq D_-f(\eta) \quad \textit{or} \quad D^-f(\xi) \leq {}_a^\xi\mathcal{K}(f) \leq D_+f(\xi).$$

Proof. Without loss of generality we may assume that $f'(a) = f'(b) = 0$, otherwise we take the function $h(x) = f(x) - xf'(a)$ for $x \in \langle a, b\rangle$. For

$$g(x) = \begin{cases} {}_a^x\mathcal{K}(f), & x \in (a, b\rangle \\ f'(a), & x = a, \end{cases} \tag{2}$$

we have $g \in \mathcal{C}\langle a, b\rangle$ and $g'(b) = -\frac{g(b)}{b-a}$. From it follows that if $g(b) > 0$, then $g'(b) < 0$, i.e., g is strictly decreasing at b. Note that also $g(a) = 0$. Since $g \in \mathcal{C}\langle a, b\rangle$, the maximum of g attains at some point $\eta \in (a, b)$ and by Proposition 4 we get $D^+g(\eta) \leq 0$ and $D_-g(\eta) \geq 0$. Immediately, from the equalities

$$D^+g(\eta) = -\frac{1}{\eta - a}\left[{}_a^\eta\mathcal{K}(f) + D^+f(\eta)\right],$$
$$D_-g(\eta) = -\frac{1}{\eta - a}\left[{}_a^\eta\mathcal{K}(f) + D_-f(\eta)\right]$$

we have the inequality $D^+f(\eta) \leq {}_a^\eta\mathcal{K}(f) \leq D_-f(\eta)$.

On the other hand, if $g(b) < 0$, then $g'(b) > 0$, so g is strictly increasing at b with $g(a) = 0$. Therefore, g attains minimum at $\xi \in (a, b)$, and $D_+g(\xi) \geq 0$ and $D^-g(\xi) \leq 0$, which implies the required inequalities.

Finally, if $g(b) = 0$, whereas $g \in \mathcal{C}\langle a, b\rangle$ and $g(a) = 0$, then g has maximum at η or minimum at ξ somewhere between points a and b. Therefore, similarly as above, either $D^+g(\eta) \leq 0$ and $D_-g(\eta) \geq 0$, or $D_+g(\xi) \geq 0$ and $D^-g(\xi) \leq 0$. This is equivalent to the assertion of theorem. □

Theorem 6 (Reich, 1969)**.** *Let $f \in \mathcal{C}\langle a, b\rangle$ and f be differentiable at b. If all four Dini's derivatives of f on (a, b) are finite and $[f(b) - f(a)]f'(b) \leq 0$, then there exist $\eta, \xi \in (a, b\rangle$ such that*

$$D^+f(\eta) \leq 0 \leq D_-f(\eta) \quad \textit{or} \quad D^-f(\xi) \leq 0 \leq D_+f(\xi). \tag{3}$$

Proof. If $f'(b) = 0$, we put $\eta = b$ and $\xi = b$ and get the desired inequalities. If $f(b) = f(a)$ and f is nonconstant, then from continuity f attains maximum at η and minimum at $\xi \in (a, b)$. By Proposition 4 we get the statement.

Assume that $[f(b) - f(a)]f'(b) < 0$. Then either $f'(b) < 0$ and $f(b) > f(a)$, or $f'(b) > 0$ and $f(b) < f(a)$. Since $f \in \mathcal{C}\langle a, b\rangle$, in the first case $f(b) > f(a)$ and f is strictly decreasing at b, so f has maximum at $\eta \in (a, b)$, and it yields the first inequality by Proposition 4. Similarly, in the second case f attains minimum at $\xi \in (a, b)$, and we get the second required inequality. □

Now we are ready to extend Trahan's sufficient condition, see Introduction, for Dini's derivatives.

Theorem 7. *Let $f \in \mathcal{C}\langle a, b\rangle$ and f be differentiable at a and b. If all four Dini's derivatives of f are finite on (a, b) and*

$$\left(f'(b) - {}_a^b\mathcal{K}(f)\right) \cdot \left(f'(a) - {}_a^b\mathcal{K}(f)\right) \geq 0,$$

then there exist $\eta, \xi \in (a, b)$ such that (3) holds.

Proof. Consider the function g given by (2). It is easy to verify that g fulfills all assumptions of Theorem 6 and substituting g in (3) we get the required result. □

2.2 Flett's theorem with v-derivative

Derivative of a function f at a point x_0 can be seen as a limit at x_0 of a proportion of an increment of function f and an increment of function $v(t) = t$. If this function v is not fixed, we come to a generalized notion of derivative with respect to another function introduced by Ohriska in [5]. In what follows we suppose real functions $f, g, v, v_1, \dots$ of (one) real variable and $\mathcal{O}^*(x_0)$ denotes a reduced neighbourhood of a point x_0.

Definition 8 (Ohriska, 1989)**.** Let functions f and v be defined in a neighbourhood $\mathcal{O}(x_0)$ of a point $x_0 \in \mathbb{R}$, and let for each $x \in \mathcal{O}^*(x_0)$ be $v(x) \neq v(x_0)$. If there exists proper limit

$$\lim_{x \to x_0} {}_{x_0}^{x}\mathcal{K}(f, v),$$

its value is called a *quasi-derivative* of f at x_0 (more precisely, derivative of f at x_0 with respect to v, resp. v-derivative of f at x_0). We denote it by $f'_v(x_0)$.

Similarly we define one-sided v-derivatives of f at x_0. Clearly, v-derivative $f'_v(x_0)$ exists if and only if there exist one-sided v-derivatives $f'^{-}_v(x_0)$ and $f'^{+}_v(x_0)$, and $f'_v(x_0) = f'^{-}_v(x_0) = f'^{+}_v(x_0)$. The set of all functions $f : M \to \mathbb{R}$ having v-derivative at each point of $M \subseteq \mathbb{R}$ for a fixed function v is denoted by $\mathcal{D}_v(M)$.

Remark 9. From Definition 8 one can see that $f'_f(x_0) = 1$ at each point x_0 such that f is defined in $\mathcal{O}(x_0)$ and condition $x \in \mathcal{O}^*(x_0)$ implies $f(x) \neq f(x_0)$.

Moreover, if v is differentiable at x_0 with $v'(x_0) \neq 0$, then $f'_v(x_0)$ exists if and only if $f'(x_0)$ exists. In this case $f'_v(x_0) = \frac{f'(x_0)}{v'(x_0)}$.

Note that strict monotonicity of f at x_0 can be obtained using v-derivative. More precisely, if v is strictly increasing at x_0 and $f'_v(x_0) > 0$, then f is strictly increasing at x_0. Also, other versions can be easily derived. Then a version of Flett's theorem with v-derivative reads as follows.

Theorem 10. *Let $v \in \mathcal{C}\langle a, b\rangle$ be strictly monotone on $\langle a, b\rangle$ and $f \in \mathcal{D}_v\langle a, b\rangle$. If $f'_v(a) = f'_v(b)$, then there exists $\eta \in (a, b)$ such that*

$$f'_v(\eta) = {}_a^{\eta}\mathcal{K}(f, v). \tag{4}$$

Proof. Without loss of generality suppose $f'_v(a) = f'_v(b) = 0$, otherwise consider the function $h(x) = f(x) - v(x)f'_v(a)$ for $x \in \langle a, b\rangle$. Put

$$g(x) = \begin{cases} {}_a^{x}\mathcal{K}(f, v), & x \in (a, b\rangle \\ f'_v(a), & x = a. \end{cases} \tag{5}$$

Clearly, $g \in \mathcal{C}\langle a, b\rangle \cap \mathcal{D}_v(a, b)$ and

$$g'_v(x) = -\frac{{}_a^{x}\mathcal{K}(f, v) - f'_v(x)}{v(x) - v(a)} = -\frac{g(x)}{v(x) - v(a)} + \frac{f'_v(x)}{v(x) - v(a)}, \quad x \in (a, b\rangle.$$

We want to show that there exists $\eta \in (a, b)$ such that $g'_v(\eta) = 0$.

Let v be strictly increasing on $\langle a, b\rangle$. By definition of g we have $g(a) = 0$. If $g(b) = 0$, then by Rolle's theorem there is $\eta \in (a, b)$ such that $g'_v(\eta) = 0$. Now let $g(b) \neq 0$. Suppose that $g(b) > 0$ (similar arguments apply in the case $g(b) < 0$). Then

$$g'_v(b) = -\frac{g(b)}{v(b) - v(a)} < 0.$$

Since $g \in \mathcal{C}\langle a, b\rangle$ and $g'_v(b) < 0$, i.e., g is strictly decreasing at b, then there exists $x_1 \in (a, b)$ such that $g(x_1) > g(b)$. From continuity of g on the interval $\langle a, x_1\rangle$ and from the inequalities $0 = g(a) < g(b) < g(x_1)$, Darboux intermediate value theorem guarantees the existence of $x_2 \in (a, x_1)$ such that $g(x_2) = g(b)$. Since $g \in \mathcal{C}\langle x_2, b\rangle \cap \mathcal{D}_v(x_2, b)$, Rolle's theorem gives $g'_v(\eta) = 0$ for some $\eta \in (x_2, b) \subset (a, b)$. The case of v being strictly decreasing on $\langle a, b\rangle$ may be proceed similarly. □

Next result eliminates the condition $f'_v(a) = f'_v(b)$. For its proof it is enough to take the function

$$\psi(x) = f(x) - {}_a^{b}\mathcal{K}(f'_v, v) \cdot \frac{(v(x) - v(a))^2}{2}, \quad x \in \langle a, b\rangle,$$

and apply the quasi-Flett's Theorem 10.

Theorem 11. *Let $v \in \mathcal{C}\langle a,b\rangle$ be strictly monotone on $\langle a,b\rangle$. If $f \in \mathcal{D}_v\langle a,b\rangle$, then there exists $\eta \in (a,b)$ such that*

$$f_v'(\eta) = {}_a^{\eta}\mathcal{K}(f,v) + {}_a^{b}\mathcal{K}(f_v',v) \cdot \frac{v(\eta)-v(a)}{2}.$$

In order to extend Pawlikowska's theorem, we need the notion of quasi-derivative of higher order.

Definition 12 (Ohriska, 1989)**.** Let $n > 1$ be a natural number. Let v_n and $f^{(n-1)}_{v_1,v_2,\dots,v_{n-1}}$ be defined in a neighbourhood $\mathcal{O}(x_0)$ of a point $x_0 \in \mathbb{R}$ and $v_n(x) \neq v_n(x_0)$ for each $x \in \mathcal{O}^*(x_0)$. If there exists proper limit

$$\lim_{x\to x_0} {}_{x_0}^{x}\mathcal{K}\left(f^{(n-1)}_{v_1,v_2,\dots,v_{n-1}}, v_n\right),$$

we call it nth *v-derivative* of function f at x_0 and we denote it $f^{(n)}_{v_1,v_2,\dots,v_n}(x_0)$.

In the case when all functions v_n are the same, i.e., $v_1 = v_2 = \dots = v_n = v$, we will write $f^{(n)}_{v^n}(x_0)$ instead of $f^{(n)}_{v_1,v_2,\dots,v_n}(x_0)$. The next result is a quasi-Pawlikowska's theorem for nth v-derivative.

Theorem 13. *Let $v \in \mathcal{C}\langle a,b\rangle$ be strictly increasing on $\langle a,b\rangle$. If $f \in \mathcal{D}_{v^n}\langle a,b\rangle$ and $f^{(n)}_{v^n}(a) = f^{(n)}_{v^n}(b)$, then there exists $\eta \in (a,b)$ such that*

$$ {}_a^{\eta}\mathcal{K}(f,v) = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i!}(v(\eta)-v(a))^{i-1} f^{(i)}_{v^i}(\eta). \tag{6}$$

Proof. The idea of proof consists in generalizing the proof of Theorem 10 using $(n-1)$th derivative of function g (suitably defined at $x=a$) and Rolle's theorem. Namely,

$$G_f(x) = \begin{cases} g^{(n-1)}_{v^{n-1}}(x), & x \in (a,b\rangle \\ \frac{1}{n} f^{(n)}_{v^n}(a), & x = a. \end{cases}$$

plays the crucial role, because $G_f \in \mathcal{C}\langle a,b\rangle \cap \mathcal{D}_{v^n}(a,b)$ and

$$g^{(n)}_{v^n}(x) = \frac{f^{(n)}_{v^n}(x)}{v(x)-v(a)} - n\frac{g^{(n-1)}_{v^{n-1}}(x)}{v(x)-v(a)}, \quad x \in (a,b\rangle.$$

This can be verified by induction. Moreover, if $f^{(n+1)}_{v^{n+1}}(a)$ exists, then

$$\lim_{x\to a^+} g^{(n)}_{v^n}(x) = \frac{1}{n+1} f^{(n+1)}_{v^{n+1}}(a).$$

The rest of the proof is analogous to the proof of Theorem 10. $\square$

A different proof is based on iteration of quasi-Flett's theorem for v-derivative applied for function

$$\varphi_k(x) = \sum_{i=0}^{k} \frac{(-1)^{i+1}}{i!}(k-i)(v(x)-v(a))^i f^{(n-k+i)}_{v^{n-k+i}}(x) + v(x) f^{(n-k+1)}_{v^{n-k+i}}(a)$$

with $k = 1, 2, \ldots, n$.

Remark 14. We may observe that the result of Theorem 13 holds for the case $v_1 = v_2 = \cdots = v_n = v$. The question is whether one can prove an analogy of Pawlikowska's theorem for different v's. However, we failed even for two different functions v_1 and v_2. The problem is that the techniques of proofs for original Pawlikowska's theorem could not be used, or they did not give any reasonable result.

In connection with v-derivative we may ask which kind of integral is its corresponding counterpart, i.e., how an integral version of quasi-Flett's theorem for v-derivative looks like. It is possible to show that it is a special kind of *Riemann-Stieltjes integral* with respect to a strictly monotone function $v \in \mathcal{C}\langle a, b\rangle$. Now the integral version of quasi-Flett's theorem takes the following form.

Theorem 15. *Let $v \in \mathcal{C}\langle a, b\rangle$ be a strictly monotone function on $\langle a, b\rangle$. If $f \in \mathcal{C}\langle a, b\rangle$ and $f(a) = f(b)$, then there exists $\eta \in (a, b)$ such that*

$$(v(\eta) - v(a))f(\eta) = \int_a^\eta f(x)\,\mathrm{d}v(x).$$

Proof. Consider the function $F(x) = \int_a^x f(t)\,\mathrm{d}v(t)$ on $\langle a, b\rangle$ and apply the quasi-Flett's theorem for v-derivative. □

In what follows we show further results for Riemann-Stieltjes integral in connection with quasi-Flett's theorem extending Tong's sufficient condition. For that purpose for functions $f, v : \langle a, b\rangle \to \mathbb{R}$, where v is continuous and strictly monotone on $\langle a, b\rangle$, we introduce

$$A_f^*(a, b) = \frac{f(a) + f(b)}{2}, \qquad I_f^*(a, b) = \frac{1}{v(b) - v(a)} \int_a^b f(x)\,\mathrm{d}v(x).$$

Theorem 16. *Let $v \in \mathcal{C}\langle a, b\rangle$ be strictly monotone on $\langle a, b\rangle$ and $f \in \mathcal{C}\langle a, b\rangle \cap \mathcal{D}_v(a, b)$.*

(i) If $A_f^*(a, b) = I_f^*(a, b)$, then there exists $\eta \in (a, b)$ such that (4) holds.

(ii) There exists $\zeta \in (a, b)$ such that

$$f_v'(\zeta) = {}_a^\zeta\mathcal{K}(f, v) + \frac{6[A_f^*(a, b) - I_f^*(a, b)]}{(v(b) - v(a))^2}(v(\zeta) - v(a)).$$

Proof. (i) Consider the function

$$h(x) = \frac{f(x) + f(a)}{2}(v(x) - v(a)) - \int_a^x f(t)\,\mathrm{d}v(t), \quad x \in \langle a, b\rangle.$$

Then $h \in \mathcal{C}\langle a,b\rangle \cap \mathcal{D}_v(a,b)$ with $h(a)=0$ and

$$h(b) = (v(b)-v(a))[A_f^*(a,b) - I_f^*(a,b)] = 0.$$

By Rolle's theorem for function h on the interval $\langle a,b\rangle$ there exists $\eta \in (a,b)$ such that $h_v'(\eta) = 0$. Since

$$h_v'(x) = \frac{1}{2} f_v'(x)(v(x)-v(a)) + \frac{1}{2}[f(x)+f(a)] - f(x), \quad x \in (a,b),$$

then from $h_v'(\eta)=0$ we have $f_v'(\eta) = {}_a^\eta\mathcal{K}(f,v)$.

(ii) Put

$$H(x) = f(x) - \frac{6[A_f^*(a,b) - I_f^*(a,b)]}{(v(b)-v(a))^2}(v(x)-v(a))(v(x)-v(b)), \quad x \in \langle a,b\rangle.$$

Then $H \in \mathcal{C}\langle a,b\rangle \cap \mathcal{D}_v(a,b)$ with $H(a) = f(a)$ and $H(b) = f(b)$. Thus, $A_H^*(a,b) = A_f^*(a,b)$ and a short computation yields $I_H^*(a,b) = A_H^*(a,b)$. Then by (i) there exists $\zeta \in (a,b)$ such that $H_v'(\zeta) = {}_a^\zeta\mathcal{K}(H,v)$. Since

$$H_v'(x) = f_v'(x) - \frac{6[A_f^*(a,b) - I_f^*(a,b)]}{(v(b)-v(a))^2}(2v(x)-v(a)-v(b)),$$

we conclude

$$\begin{aligned} & f_v'(\zeta) - \frac{6[A_f^*(a,b) - I_f^*(a,b)]}{(v(b)-v(a))^2}(2v(\zeta)-v(a)-v(b)) \\ &= \frac{1}{v(\zeta)-v(a)}\left[f(\zeta) - \frac{6[A_f^*(a,b) - I_f^*(a,b)]}{(v(b)-v(a))^2}(v(\zeta)-v(a))(v(\zeta)-v(b)) - f(a)\right], \end{aligned}$$

which is equivalent to the desired statement. $\square$

2.3 Flett's theorem with symmetric derivative

Derivative of a real function $f:\mathbb{R}\to\mathbb{R}$ is a function $f':\mathbb{R}\to\mathbb{R}$ that is defined at a point x by the limit

$$f'(x) = \lim_{h\to 0} {}_x^{x+h}\mathcal{K}(f)$$

whenever this limit exists. A small modification has suprising consequences.

Definition 17. Let f be defined on a neighbourhood $\mathcal{O}(x_0)$ of a point x_0. We say that f has *symetric derivative* at x_0, if there exists a proper limit $\lim\limits_{h\to 0} {}_{x_0-h}^{x_0+h}\mathcal{K}(f)$. The value of this limit is denoted by $f^s(x_0)$.

If f has symmetric derivative at each point of a set M, we write $f \in \mathcal{D}^s(M)$. Clearly, $\mathcal{D}(M) \subset \mathcal{D}^s(M)$, however the reverse inclusion does not hold. More interestingly, symmetric derivative can exist at a point where the function is not defined, e.g. for $g(x) = x^{-2}$ at $x_0 = 0$ we have $g^s(0) = 0$. Consequently, $\mathcal{D}^s(M) \not\subset \mathcal{C}(M)$.

Since many results of differential calculus fail for symmetric derivative, we will use the following useful result, see [1].

Lemma 18 (Aull, 1967)**.** *Let $f \in \mathcal{C}\langle a, b\rangle \cap \mathcal{D}^s(a, b)$.*

(i) If $f(b) > f(a)$, then there exists $\eta \in (a, b)$ such that $f^s(\eta) \geq 0$.

(ii) If $f(b) < f(a)$, then there exists $\xi \in (a, b)$ such that $f^s(\xi) \geq 0$.

Proof. We prove only (i), the second statement can be proved analogously. Let k be a real number such that $f(a) < k < f(b)$. Then the set $M = \{x \in \langle a, b\rangle; f(x) > k\}$ is non-empty and bounded from below by a. Since $M \subset \mathbb{R}$, there exists $\eta = \inf M$. From $f \in \mathcal{C}\langle a, b\rangle$ and $f(a) < k < f(b)$ we have $\eta \notin \{a, b\}$. Let $\mathcal{O}_h(\eta) = (\eta - h, \eta + h)$ be an arbitrary neighbourhood of η in the interval $\langle a, b\rangle$. Then there exist $x \in \mathcal{O}_h(\eta)$ such that $f(x + h) > k$ and $f(x - h) \leq k$. Thus,

$$f^s(\eta) = \lim_{h\to 0} \frac{f(\eta + h) - f(\eta - h)}{2h} \geq 0. \qquad \square$$

The following result can be understood as a version of Rolle's theorem for symmetric derivative.

Theorem 19 (Aull, 1967)**.** *Let $f \in \mathcal{C}\langle a, b\rangle \cap \mathcal{D}^s(a, b)$. If $f(a) = f(b)$, then there exist points $\eta, \xi \in (a, b)$ such that $f^s(\eta) \geq 0 \geq f^s(\xi)$.*

Proof. Suppose that $f(a) = 0 = f(b)$. If it is not the case, we take $h(x) = f(x) - f(a)$ on $\langle a, b\rangle$. For $f \equiv 0$ the result holds trivially, therefore we further suppose $f \not\equiv 0$. Since $f \in \mathcal{C}\langle a, b\rangle$ and $f(a) = f(b) = 0$, then there exist points $x_1, x_2 \in (a, b)$ such that

$$\begin{gathered}(f(x_1) > 0 \wedge f(x_2) < 0) \vee (f(x_1) < 0 \wedge f(x_2) > 0) \\ \vee (f(x_1) > 0 \wedge f(x_2) > 0) \vee (f(x_1) < 0 \wedge f(x_2) < 0).\end{gathered}$$

Let us consider only the case $f(x_1) > 0$ and $f(x_2) < 0$, other cases are analogous. Applying Lemma 18 on the interval $\langle a, x_1\rangle$ we get $f^s(\eta) \geq 0$ for some $\eta \in (a, x_1) \subset (a, b)$. Applying Lemma 18 again for function f on the interval $\langle a, x_2\rangle$ we conclude $f^s(\xi) \leq 0$ for some $\xi \in (a, x_2) \subset (a, b)$. $\square$

Remark 20. Simeon Reich in the paper [8] gave another sufficient condition for validity of quasi-Rolle's theorem considering differentiability of f at b and inequality $[f(b) - f(a)]f'(b) \leq 0$ instead of equality $f(a) = f(b)$. In fact, it is a version of Trahan's condition for symmetric derivative.

Next result is a quasi-Lagrange theorem for symmetric derivative.

Theorem 21 (Aull, 1967)**.** *Let $f \in \mathcal{C}\langle a,b\rangle \cap \mathcal{D}^s(a,b)$. Then there exist points $\eta, \xi \in (a,b)$ such that*

$$f^s(\eta) \leq {}_a^b\mathcal{K}(f) \leq f^s(\xi).$$

Proof. For function $g : \langle a,b\rangle \to \mathbb{R}$ defined by

$$g(x) = f(x) - f(a) - {}_a^b\mathcal{K}(f)(x-a)$$

we have $g \in \mathcal{C}\langle a,b\rangle \cap \mathcal{D}^s(a,b)$ and $g(a) = g(b) = 0$. Applying Theorem 19 we get $g^s(\eta) \leq 0 \leq g^s(\xi)$ for some $\eta, \xi \in (a,b)$, which is equivalent to $f^s(\eta) \leq {}_a^b\mathcal{K}(f) \leq f^s(\xi)$. □

Now we can state some generalizations of Flett's mean value theorem with symmetric derivative. In what follows, for f on $\langle a,b\rangle$ we use the convention $f'(a) = f^s(a)$ and $f'(b) = f^s(b)$. For $f \in \mathcal{C}\langle a,b\rangle \cap \mathcal{D}^s(a,b)$ that is differentiable at a, b we consider the function g given by (5). Clearly, $g \in \mathcal{C}\langle a,b\rangle \in \mathcal{D}^s(a,b\rangle$ with

$$g^s(x) = -\frac{f(x)-f(a)}{(x-a)^2} + \frac{f^s(x)}{x-a}, \quad x \in (a,b\rangle.$$

Since

$$[g(b)-g(a)]g'(b) = \frac{-1}{b-a}\left(f'(b) - {}_a^b\mathcal{K}(f)\right)\cdot\left(f'(a) - {}_a^b\mathcal{K}(f)\right),$$

under the condition

$$\left(f'(b) - {}_a^b\mathcal{K}(f)\right)\cdot\left(f'(a) - {}_a^b\mathcal{K}(f)\right) \geq 0$$

this product is nonpositive. By Reich's result from Remark 20 there exist points $\eta, \xi \in (a,b\rangle$ such that $g^s(\xi) \leq 0 \leq g^s(\eta)$. Thus, we have proved the following result.

Theorem 22. *Let $f \in \mathcal{C}\langle a,b\rangle \cap \mathcal{D}^s(a,b)$. If f is differentiable at points a, b and*

$$\left(f'(b) - {}_a^b\mathcal{K}(f)\right)\cdot\left(f'(a) - {}_a^b\mathcal{K}(f)\right) \geq 0, \tag{7}$$

then there exist points η, $\xi \in (a,b\rangle$ such that

$$f^s(\eta) \geq {}_a^\eta\mathcal{K}(f) \quad \textit{and} \quad f^s(\xi) \leq {}_a^\xi\mathcal{K}(f).$$

Now we eliminate the condition (7) from quasi-Flett's theorem.

Theorem 23. *Let $f \in \mathcal{C}\langle a,b\rangle \cap \mathcal{D}^s(a,b)$. If f is differentiable at points a, b, then there exist $\eta, \xi \in (a,b)$ such that*

$$f^s(\eta) \geq {}_a^\eta\mathcal{K}(f) + {}_a^b\mathcal{K}(f')\cdot\frac{\eta-a}{2} \quad \textit{and} \quad f^s(\xi) \leq {}_a^\xi\mathcal{K}(f) + {}_a^b\mathcal{K}(f')\cdot\frac{\xi-a}{2}. \tag{8}$$

Proof. Consider the function

$$\psi(x) = f(x) - {}_a^b\mathcal{K}(f') \cdot \frac{(x-a)^2}{2}, \quad x \in \langle a, b\rangle.$$

Then $\psi \in \mathcal{D}^s(a,b)$ and

$$\psi^s(x) = f^s(x) - {}_a^b\mathcal{K}(f') \cdot (x-a).$$

Since f is differentiable at the end-points of the interval $\langle a,b\rangle$, the function ψ is differentiable at the end-points of $\langle a,b\rangle$ and $\psi'(b) = \psi'(a)$. Then the result follows from quasi-Flett's Theorem 22 for function ψ on the interval $\langle a,b\rangle$. □

Now we mention a Cauchy version of quasi-Flett's theorem for symmetric derivative due to Reich [8] and we provide its interesting corollaries.

Theorem 24 (Reich, 1969)**.** *Let $f, g \in \mathcal{C}\langle a,b\rangle \cap \mathcal{D}^s(a,b)$ and f, g be differentiable at points a, b with $g'(a) \neq 0$. If $g(x) \neq g(a)$ for each $x \in (a,b\rangle$ and*

$$\left(\frac{f'(a)}{g'(a)} - {}_a^b\mathcal{K}(f,g)\right)\Big(f'(b)(g(b)-g(a)) - (f(b)-f(a))g'(b)\Big) \geq 0, \qquad (9)$$

then there exist points $\eta, \xi \in (a,b\rangle$ such that

$$[g(\eta)-g(a)]f^s(\eta) \geq [f(\eta)-f(a)]g^s(\eta) \ \text{ and } \ [g(\xi)-g(a)]f^s(\xi) \leq [f(\xi)-f(a)]g^s(\xi).$$

Proof. Define the function $h : \langle a,b\rangle \to \mathbb{R}$ by

$$h(x) = \begin{cases} {}_a^x\mathcal{K}(f,g), & x \in (a,b\rangle \\ \frac{f'(a)}{g'(a)}, & x = a. \end{cases} \qquad (10)$$

Then $h \in \mathcal{C}\langle a,b\rangle \cap \mathcal{D}^s(a,b)$ and h is differentiable at b. For its symmetric derivative holds

$$h^s(x) = -\frac{{}_a^x\mathcal{K}(f,g)}{g(x)-g(a)} g^s(x) + \frac{f^s(x)}{g(x)-g(a)}, \quad x \in (a,b).$$

By the assumption (9) we have $[h(b)-h(a)]h'(b) \leq 0$. Then by Reich's result from Remark 20 we get $h^s(\eta) \geq 0 \geq h^s(\xi)$, which is equivalent to the desired result. □

Corollary 25. Let $f, g \in \mathcal{C}\langle a,b\rangle \cap \mathcal{D}^s(a,b)$ and f, g be differentiable at points a, b. Let $g'(a) \neq 0$, $g'(b) > 0$ and $g(x) \neq g(a)$ for each $x \in (a,b\rangle$. If

$$\frac{f'(a)}{g'(a)} = \frac{f'(b)}{g'(b)}, \qquad (11)$$

then there exist points $\eta, \xi \in (a,b)$ such that

$$[g(\eta)-g(a)]f^s(\eta) \geq [f(\eta)-f(a)]g^s(\eta) \ \text{ and } \ [g(\xi)-g(a)]f^s(\xi) \geq [f(\xi)-f(a)]g^s(\xi).$$

Proof. Consider the function h given by (10) and investigate two cases.

(i) Suppose that $[g(b)-g(a)]f'(b)=[f(b)-f(a)]g'(b)$. Then

$$h(b)-h(a)=\frac{f(b)-f(a)}{g(b)-g(a)}-\frac{f'(a)}{g'(a)}=\frac{f'(b)}{g'(b)}-\frac{f'(a)}{g'(a)}=0.$$

Applying Theorem 21 for function h on the interval $\langle a,b\rangle$ there exist points $\eta,\xi\in(a,b\rangle$ such that $h^s(\eta)\geq 0\geq h^s(\xi)$ which yields the desired result.

(ii) Further let us consider the case $[g(b)-g(a)]f'(b)\neq[f(b)-f(a)]g'(b)$. Therefore, either

$$[g(b)-g(a)]f'(b)-[f(b)-f(a)]g'(b)>0 \tag{12}$$

or

$$[g(b)-g(a)]f'(b)-[f(b)-f(a)]g'(b)<0. \tag{13}$$

Then it holds

$$f'(b)-\frac{f(b)-f(a)}{g(b)-g(a)}g'(b)>0,$$

and by (11) we have

$$\frac{f'(a)}{g'(a)}g'(b)-\frac{f(b)-f(a)}{g(b)-g(a)}g'(b)>0. \tag{14}$$

From the inequality (12) and (14) we get

$$\left(\frac{f'(a)}{g'(a)}g'(b)-\frac{f(b)-f(a)}{g(b)-g(a)}g'(b)\right)\Big([g(b)-g(a)]f'(b)-[f(b)-f(a)]g'(b)\Big)>0.$$

Since $g'(b)>0$, we conclude

$$\left(\frac{f'(a)}{g'(a)}-\frac{f(b)-f(a)}{g(b)-g(a)}\right)\Big[g(b)-g(a)]f'(b)-[f(b)-f(a)]g'(b)\Big)>0.$$

We proceed analogically in the case of inequality (13). Then we apply Theorem 24. It is easy to see that $\eta\neq b$ and $\xi\neq b$ which completes the proof. $\square$

Much more interesting is an observation that from Theorem 24 we may prove the original Flett's theorem.

A new proof of original Flett's theorem. Let $f\in\mathcal{D}\langle a,b\rangle$ with $f'(a)=f'(b)$ and put $g(x)=x$. Consider the function

$$h(x)=\begin{cases}{}_a^x\mathcal{K}(f), & x\in(a,b\rangle\\ f'(a), & x=a.\end{cases}$$

Clearly, $h\in\mathcal{C}\langle a,b\rangle\cap\mathcal{D}(a,b)$. If $f'(b)={}_a^b\mathcal{K}(f)$, then $h(a)=h(b)$ and by Rolle's theorem there exists $\eta\in(a,b)$ such that $h'(\eta)=0$, which is equivalent to (1).

Now, let $f'(b) \neq {}_a^b\mathcal{K}(f)$. Then by Theorem 24 there exist points $\eta, \xi \in (a, b)$ such that $f'(\eta) \geq {}_a^\eta\mathcal{K}(f)$ and $f'(\xi) \leq {}_a^\xi\mathcal{K}(f)$. If we have equality in both inequalities, we have the desired result. If both inequalities are strict, we put

$$j(x) = f'(x) - {}_a^x\mathcal{K}(f), \quad x \in \langle \eta, \xi \rangle \text{ or } \langle \xi, \eta \rangle.$$

Then there exists a function J such that $J'(x) = j(x)$ on $\langle \eta, \xi \rangle$, so j is a darboux function on $\langle \eta, \xi \rangle$. Since $j(\eta) > 0 > j(\xi)$, then there exists a point $\zeta \in (\eta, \xi) \subset \langle a, b \rangle$ such that $j(\zeta) = 0$. $\square$

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Bibliography

[1] C. F. Aull: *The First Symmetric Derivative*. Amer. Math. Monthly 74, (1967), 708–711.

[2] T. M. Flett: *A Mean Value Theorem*. Math. Gazette 42, (1958), 38–39.

[3] T. V. Lakshminarasimhan: *A Mean Value Theorem - an Extension*. Amer. Math. Monthly 73, (1966), 862–863.

[4] J. Molnárová, O. Hutník: *On Flett's Mean Value Theorem*. Aequationes Math. 89(2), (2015), 1133–1165.

[5] J. Ohriska: *Oscillation of Differential Equations and v-derivatives*. Czechoslovak Math. J. 39(114), (1989), 24–44.

[6] I. Pawlikowska: *An Extension of a Theorem of Flett*. Demonstratio Math. 32, (1999), 281–286.

[7] T. Riedel, P. K. Sahoo: *Mean Value Theorems and Functional Equations*. World Scientific, River Edge, NJ, 1998.

[8] S. Reich: *On Mean Value Theorems*. Amer. Math. Monthly 76, (1969), 70–73.

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Using GeoGebra for solving equations and inequalities

Jozef Doboš[1]

Abstract: In this article, we show how to use GeoGebra for solving equations and inequalities in school Mathematics. Many solutions offered by GeoGebra in its CAS View are not too dissimilar from the answers expected in a school context. However, there are multiple differences that could confuse both students and teachers. Using actual worked examples, the aim of this article is to point out some of the reasons for these differences and to demonstrate how even in such situations GeoGebra can deliver solutions expected in school Mathematics.

Keywords: GeoGebra; Equations; Inequalities.
Mathematics Subject Classification: 97U70.

1 Introduction

The GeoGebra software is a powerful tool for solving equations and inequalities in school Mathematics. Many solutions offered by this tool are similar to the answers expected in a school context. However, there are multiple differences that could confuse both students and teachers. The aim of this article is to highlight several reasons for these discrepancies. We also show how even in such situations solutions expected in school Mathematics can be delivered via GeoGebra.

2 Solving some equations and inequalities by a software

Equations and inequalities are to be solved in the computer algebra window (CAS View). In order to effectively use this tool, students and teachers need to understand the sources of unexpected solutions and have an idea how to deal with them.

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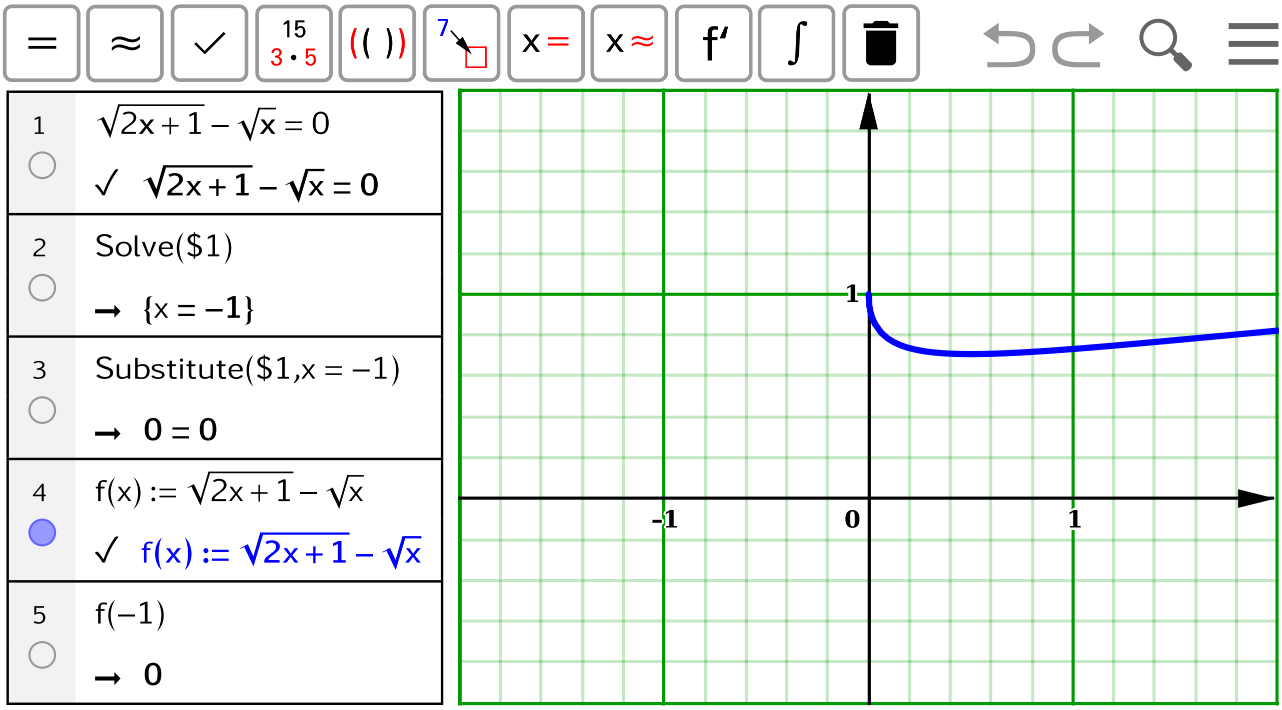

Figure 1: Solving the equation $\sqrt{2x+1} - \sqrt{x} = 0$

The first example is a solution to equation

$$\sqrt{2x+1} - \sqrt{x} = 0. \tag{1}$$

Put equation (1) into cell No. 1 either using a virtual keyboard or by directly typing sqrt(2x+1)-sqrt(x)=0 in there (see Figure 1). Next, using a mouse, click the $\boxed{\checkmark}$ button (third left). This ensures the insertion of an expressions without (not always correct) changes.

Write Solve($1) into cell No. 2, where Solve(<equation>) is a command for solving equations, and $1 is a reference to the contents of a cell No. 1. Afterwards, click the $\boxed{=}$ button in the top toolbar (first left). GeoGebra will print out a list of solutions to this equation, hence the angle brackets. GeoGebra is trying to persuade us that equation (1) has exactly one real root $x = -1$.

To check this solution, we can substitute it back into the equation saved in a cell No. 1. Thus, write Substitute($1,x=-1) into cell No. 3. Next, click the $\boxed{=}$ button in the top toolbar (first left). The result is $0 = 0$, verifying that $x = -1$ is a valid solution (by GeoGebra). However, we know that $x = -1$ cannot be a solution to (1) since negative numbers cannot appear under a square root sign. This is what students are thought in secondary schools. Something not being correct is even shown by GeoGebra itself in the graphical window. Write f(x):=sqrt(2x+1)-sqrt(x) in a cell No. 4. That is how in CAS window one defines a function f using $f(x) = \sqrt{2x+1} - \sqrt{x}$.
Next, click the $\boxed{\checkmark}$ button in the top toolbar (third left). GeoGebra will plot a

graph of a function f, what is actually the left-hand side of equation (1).

Write f(-1) in a cell No. 5 in order to check whether GeoGebra can calculate the value of function f at $x = -1$. Next, click the $\boxed{=}$ button in the top toolbar (first left). GeoGebra is trying to persuade us that $f(-1) = 0$. However, that would mean that the point $(-1, 0)$ lies on a graph of the function f, but it does not. There is clearly a discrepancy.

An explanation is found when we solve the equation

$$\sqrt{2x+1} = \sqrt{x} \tag{2}$$

instead of equation (1).

Actually, equation (1) came from equation (2) which can be found in [5]. That is a dissertation which deals on solving equations in school Mathematics using computer algebra systems (CAS). The substitution in a cell No. 3 shows that after plugging $x = -1$ into equation (2), equality ί = ί is obtained, where ί is a special symbol used by GeoGebra for imaginary values. This must not to be confused with the letter í. However, we can insert it directly using keyboard shortcut Alt+i, or via a virtual keyboard. See Figure 2.

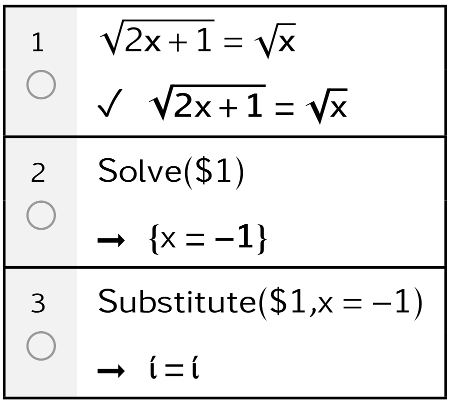

Figure 2: Solving the equation $\sqrt{2x+1} = \sqrt{x}$

This way of solving irrational equations is not specific to GeoGebra. For instance, commercial software Maple behaves in the exact same manner. The only difference is in the labelling of the imaginary values as can be seen in an example on Figure 3. CAS systems usually calculate the principal root (главное значение корня), cf. [4] and [1]. If non-zero complex number z is expressed in a trigonometric form $z = r(\cos\theta + \mathrm{i}\sin\theta)$, where $-\pi < \theta \leq \pi$, then the principal value of n-root of z is a complex number $w_0 = \sqrt[n]{r}\left(\cos\frac{\theta}{n} + \mathrm{i}\sin\frac{\theta}{n}\right)$.

$$solve\left(\sqrt{2x+1}-\sqrt{x}=0\right)$$

$$-1$$

$$f := x \to \sqrt{2x+1}$$

$$x \to \sqrt{2x+1}$$

$$f(-1)$$

$$I$$

$$g := x \to \sqrt{x}$$

$$x \to \sqrt{x}$$

$$g(-1)$$

$$I$$

Figure 3: Solving the equation $\sqrt{2x+1}-\sqrt{x}=0$ via Maple

For GeoGebra to solve equation (2), as would be expected in school Mathematics, the domain of the equation has to be taken into account, too. That can be obtained by solving a system of inequalities $2x+1 \geq 0$, $x \geq 0$. It is sufficient in a cell No. 1 in the CAS window to type

```
Solve({sqrt(2x+1)=sqrt(x),2x+1>=0,x>=0})
```

By doing so, we solve a mixed system; one equation and two inequalities (that define a domain of the equation). Next, click $\boxed{=}$ button in the top toolbar (first right). GeoGebra now prints out a correct result – an empty list in a form of `{}`.

We can also try the suggested procedure on the following equation

$$\sqrt{2x^2+5x+1}=\sqrt{x+1}. \tag{3}$$

Command `Solve(sqrt(2x^2+5x+1)=sqrt(x+1)` results in a list of solutions `{x=-2,x=0}`, out of which $x=-2$ does not belong to the domain of equation (3). However, command

```
Solve({sqrt(2x^2+5x+1)=sqrt(x+1),2x^2+5x+1>=0,x+1>=0}
```

results in an expected list of solutions `{x=0}`.

Interesting is also the situation with a cube root. Our secondary school textbooks explicitly say that we do not define n-th root of negative numbers. See for instance [3]. Only in textbook [2], it is mentioned that it is possible to define a cube root of any (even negative) number. But immediately in the next sentence it is said that we are only interested in roots of positive numbers and indeed it is being followed. Therefore, when solving equation

$$\sqrt[3]{24+x}+\sqrt{12-x}=6 \tag{4}$$

one must first decide whether to allow for a cube root of negative numbers or not. GeoGebra will otherwise include also number $x=-88$ among the solutions.

Similarly to the above, in program GeoGebra, one is solving a mixed system consisting of an equation and one or two inequalities determining the domain of that equation.

The difference between buttons $\boxed{\checkmark}$ and $\boxed{=}$ can be tested, for instance, using equation

$$\sqrt{x-1}\sqrt{x-2} = 1. \tag{5}$$

GeoGebra struggles with equations that have infinitely many solutions. As an example, try to solve the equation

$$\sqrt{x-1} = 1 + \sqrt{x - 2\sqrt{x-1}}. \tag{6}$$

A solution of equation (6) is any real number $x \geq 2$. This can be verified via a graph of the function $f(x) = 1 + \sqrt{x - 2\sqrt{x-1}} - \sqrt{x-1}$. See Figure 4.

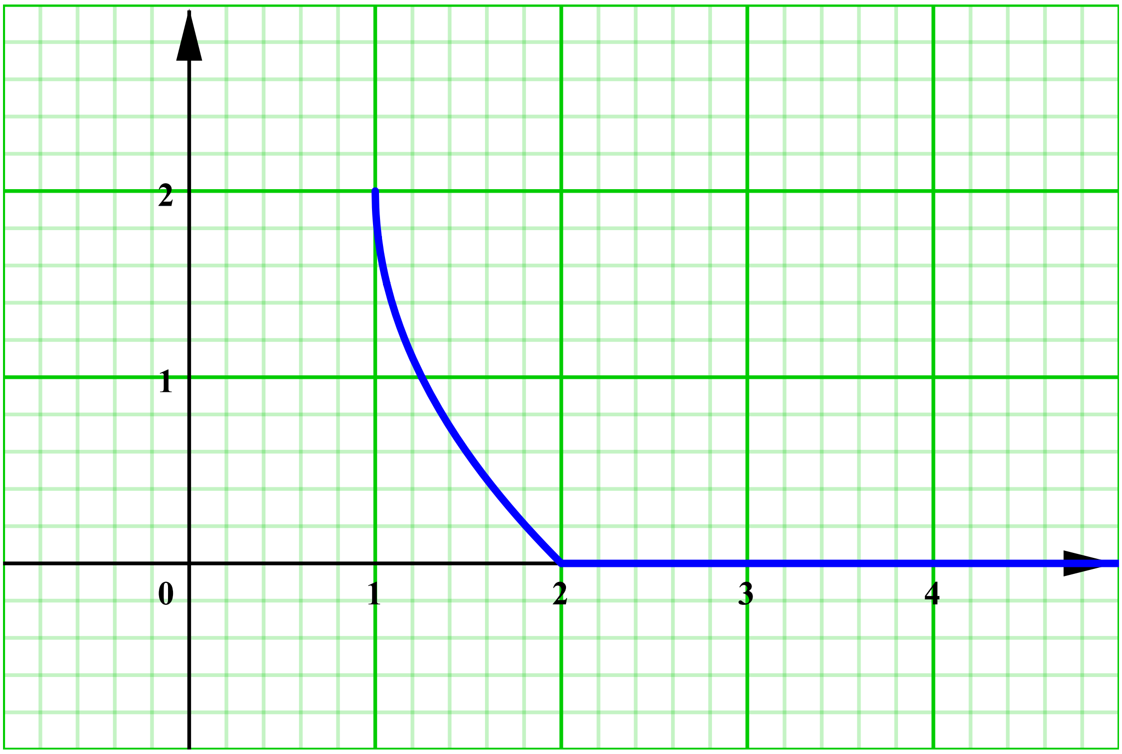

Figure 4: The graph of the function $f(x) = 1 + \sqrt{x - 2\sqrt{x-1}} - \sqrt{x-1}$

One needs to be careful also when solving equations with parameters since GeoGebra does not offer a discussion regarding the parameters. As an example, try to solve equation

$$\frac{1}{x+a+b} = \frac{1}{x} + \frac{1}{a} + \frac{1}{b}. \tag{7}$$

GeoGebra returns roots $x = -a$, $x = -b$. Even when checking solutions, it attempts to persuade us that it is correct. However, it is sufficient to move everything to the right-hand side and then factoring it, for example use command

Factor(0=1/x+1/a+1/b-1/(x+a+b)). After this arrangement, a discussion regarding the parameters can be achieved relatively easily.

When dealing with logarithmic equations and inequalities, it is also recommended to solve mixed systems, similarly to how it was done with irrational equations. Try to solve equation

$$\log_{10}(x-2) - \log_{10}(x-1) = 1. \tag{8}$$

Sometimes it is needed to help oneself with a command Simplify. For instance, when solving inequality

$$\left(\frac{1}{2}\right)^{\log_2(x^2-1)} < 1 \tag{9}$$

using command Solve({Simplify((1/2)^(ld(x^2-1))<1),x^2-1>0},x) one gets $\{-\sqrt{2} > x, x > \sqrt{2}\}$. That is the list of intervals the union of which defines the solution set for inequality (9).

A problem with domain also appears when solving trigonometric equations, for instance

$$\frac{1-\cos(2x)}{\sin(2x)} = 0. \tag{10}$$

Equation (10) is solved using command Solve({(1-cos(2x))/sin(2x)=0,sin(2x)!=0},x).

Even bigger problem GeoGebra encounters with equation

$$\sqrt{\cos x - 1} = 0. \tag{11}$$

This is because the domain of equation (11) consist of isolated points. Equation (11) is solved using command Solutions(sqrt(cos(x)-1)=0).

3 Conclusion

GeoGebra is not perfect and that is fine. It will not do everything instead of us. To get the correct answers, one must first know something about Mathematics and also needs to know how to use it.

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Bibliography

[1] И. И. Бронштейн, К. А. Семендяев: *Справочник по математике для инженеров и учащихся втузов*, «Наука», Москва, 1986. (In Russian)

[2] Z. Kubáček: *Matematika pre 2. ročník gymnázií a 6. ročník gymnázií s osemročným štúdiom*, druhá časť, Orbis Pictus Istropolitana, Bratislava, 2010. (In Slovak)

[3] O. Odvárko, M. Božek, M. Ryšánková, J. Smida: *Matematika pre 2. ročník gymnázia*, SPN, Bratislava, 1985. (In Slovak)

[4] И. И. Яцкин: *Алгебра. Теоремы и алгоритмы, учеб. пособие*, Иван. гос. ун-т, Иваново, 2006. (In Russian)

[5] E. Tõnisson: *Differences Between Expected Answers and the Answers Offered by Computer Algebra Systems to School Mathematics Equations*, Dissertationes Mathematicae Universitatis Tartuensis 122, University of Tartu Press, 2017.

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The usage of graphs in economics

Erika Fecková Škrabuľáková[1] **and Elena Grešová**[2]

Abstract: In this paper we were interested in the usage of graphs and graph-like structures in graph theory means in economics. We discuss their utilization in many different situations and point out to the row of economics related problems that found their solutions via applications of graph theory tools or concatenation of graph theory aids and other tools of mathematics.

Although the aim of this small overview contribution is to highlight the width range of applications of graph structures in economics, we also point out to their other applications outside mathematics.

Keywords: Application; Economics; Graph.

Mathematics Subject Classification: 05C90, 94C15.

1 Introduction

Mathematics and economics are closely related. In order to acquire general knowledge of economics or financial literacy, the wisdom from mathematics is necessary [6]. The results of several studies confirm that the importance of mathematics both for study economics and solve practical tasks of economy grows [16], [1]. Beside mathematical statistics, actuarial mathematics and associative calculus, one of the strong branches of mathematics with widespread utilization in economics is the graph theory.

In this paper we aim to present a short overview on usage of graphs and graph-like structures in economics. The graphs are understood as structures in graph theory means - representations of set of points - vertices, and a set of curves - edges.

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A graph G is understood as an ordered pair $G = (V(G), E(G))$ of a finite set $V(G) = \{v_1, v_2, \ldots, v_n\}$ of vertices, and a set $E(G) = \{e_1, e_2, \ldots, e_n\}$ of unordered pairs of vertices - the set of edges. The relative positions of points representing vertices and curves representing the edges have no significance [4].

2 The versatility of graph theory

Graph structures called trees are very often used in economics. Trees are defined as connected graphs on n vertices and $n-1$ edges and a graph G is said to be connected iff there exists a path - the sequence of different consecutive vertices and edges, between every two vertices of G. Closed path or a cycle C_n $(n > 3)$ is a graph structure of consecutive n vertices and n edges where the last vertex is identical to the first one. A simple graph that does not contain any cycle C_n $(n > 3)$ as a subgraph is called acyclic. Hence, every tree is an acyclic graph.

For classification and prediction in economics are often used decision trees (see e. g. [24], [10], [31]). An example of a decision tree used for a mortgage application assessment (see [6]) can be find at Figure 1. Similar trees are used to predict, analyse and explain the relationship between measurements about some item and its target value. They can be used as simple models that allow automatic learning in which costs can be also included. Graph algorithms allow automatically prune these trees and their complexity can be controlled [24]. Thanks to simple representation, the decision trees offer high classification accuracy in a simple, reliable and effective way.

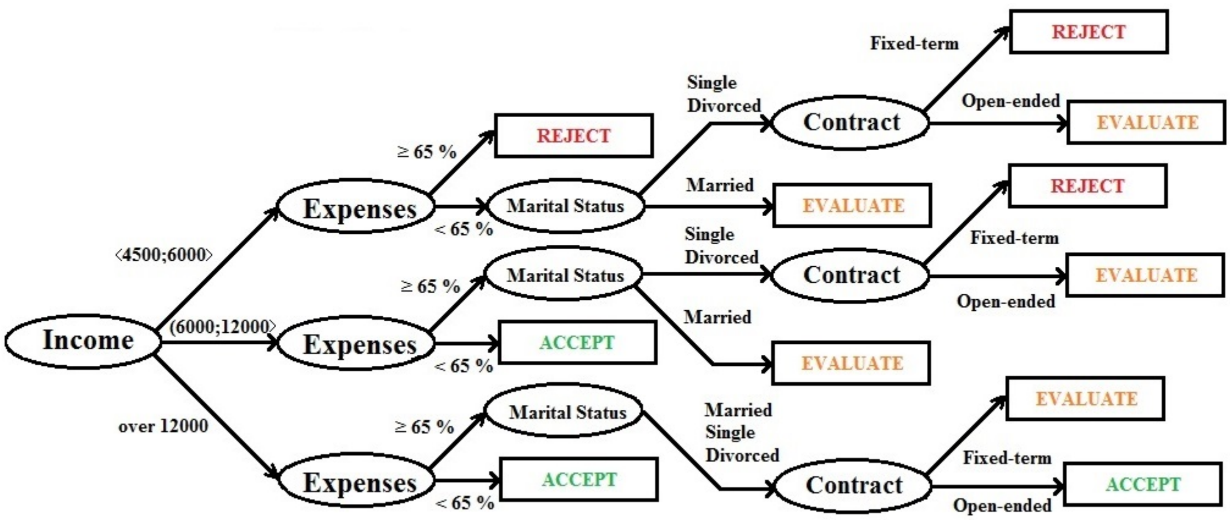

Figure 1: Decision tree used for the mortgage application assessment

Mediating the application of trees as graph structures in economics, the whole graph representation of a problem has not necessary being interpreted as a tree (or a forest of trees). Sometimes it is searched for subgraphs with prescribed features some of which classify these subgraphs into the family of trees. For example the minimum spanning tree was utilized for portfolio analysis in [25]. The dynamic asset tree was created from the correlation matrix consisting of stock returns. In [25] a complex and worthwhile economic system representing the stock market was established. The progression of the proposed tree was described in a time horizon. During a stock market crisis its retraction was clearly observed.

For a representation of probability distribution over a set of random variables the Bayesian dependence networks can be used. Bayesian dependence networks are represented by directed acyclic graphs. Directed graph or a digraph is a graph where each edge has an assigned orientation. Formally, a digraph D is an ordered triple $(V(D), E(D), \psi_D)$, where $V(D)$ is a nonempty set of vertices, $E(D)$ is a set of arcs (disjoint from $V(D)$) and ψ_D is an incidence function that associates with each arc of D an ordered pair of (not necessarily distinct) vertices from $V(D)$. (For the notation and graph theory terminology we refer to [4].)

Considering the Bayesian dependence networks, the vertices of these graphs correspond to domain variables, while the arcs defining a set of independence assumptions. They have many applications in economics. For example, the Bayesian classification technique and ordered regressions were used for the analysis of work-related stress determinants and their comparison in cross-cultural context in [2].

Graph theory found its applications considering commodity markets, as well. For instance, the particularly market integration with support of graph theory has been studied in [20] where the systemic risk was analysed. Authors took into account twelve years long time period in which on the basis of daily futures returns of the specific market bonds were inspected. These bonds were related to the particular commodities as well as the commodity complex and other assets with financial background.

The implementation of graphs in business was confirmed by Haigh and Bessler in [17] over their research tackling price discovery between spatially separated commodity markets and the transportation market connecting them with each other. Findings pointed out that transportation rates were crucial in the price discovery procedure lending support for the movements concerning exchange traded barge rate futures contracts.

The methodology based on causal analysis with, inter alia, graph theory grounding was proposed by [21]. The presented approach served for more feasible exploration of the economy's sectors. The point was to simplify the identification of the relationships between variables that constitute particular sector. The development lied in enabling the transformation of a great number of data into a directed graph.

The big advantage of the graph theory is that thanks to its simple yet powerful aids we are able to better understand the variables and their relative interdependencies. This is often used in a risk mitigation. The risk mitigation environment can be, for example, quantified and presented via a numerical index and studied via graph theory tools accompanied by matrix methods (see e. g. [8]).

The relation between the political uncertainty and the firms' financing decisions was studied in [22]. Authors of this paper established a new qualitative on heuristics-based model and derived a transitional graph to study all possible past/future behaviour of the examined variables.

Markets and systemic risk were investigated in [32], as well. The financial system was interpreted here as a graph in which each edge is given a numerical weight which typify an extensive network of cooperating agents and flow of information. Daily returns of selected stocks were examined in a fifteen year period. The focus was put on evidence about forming a financial "crash hallmark" by matching changes in Ricci curvature.

The input-output analysis represents an extensive area of the economics where graph theory is applied, as well. Some conventional concepts related to stated analysis were examined through research of [11]. Specifically, autonomous sets, indecomposable matrices and fundamental products. Selected suggestions on graph theory were adjusted and put in an application for the given concepts. This investigation comprised several steps: digraph definition linked to the economic system, reflection concerning graph's topological properties with regard on mentioned economical concepts and algorithms suggestion. What is more, there appeared also an instance from a real-life economic practice.

Another study dedicated to the input-output analysis in the graphs context was prepared by Halkos and Tsilika in [15]. Authors scrutinized trade activities pertaining to numerous sectors and regions from the networks point of view. The enforcement was put on reparation of the structure architecture connected to the interrelations. The outputs visualization was realized for the purpose to illustrate the quantity of trade links, the interaction of the trading partners as well as the density of interrelations. Assessing the vertex degrees was performed with regard on measuring the density and connectivity.

Graphs serve a simple yet worth tool for exploring graph-based structural patterns, linkages and their topological properties [34]. The dependence between different economic indicators and their effect on countries economy [10], the relations inside and outside distinct organizations [14] as well as the structure of the control network of transnational corporations and effects on the global market competition [35] and many other similar tasks [23] can be modelled and studied based on the topological structure of respective graph models. These relations (represented by edges of a graph) between the objects of interests (represented by vertices of the graph) may not be only highlighted but also studied and analysed based on graph colourings. A k-edge colouring of a graph $G(V(G), E(G))$ can be understand as a partition $(E_1, E_2, \ldots, E_k)$ of the set $E(G)$, where E_i denotes the (possibly empty) subset of $E(G)$ assigned colour i [4].

An edge-colouring of digraphs was used e.g. by processing of economic data with a view to their subsequent comparison in [10]. Authors discuss the relation between the colouring and incidence function of oriented graphs with the correlation among the economic parameters and show that graphs are worth to derive gathered knowledge of economic information. Graphs are used here in order to compare macroeconomic parameters for selected countries (see Figure 2) such as the gross domestic product, quality of life, EU direct investment positions, flows and income, and net export.

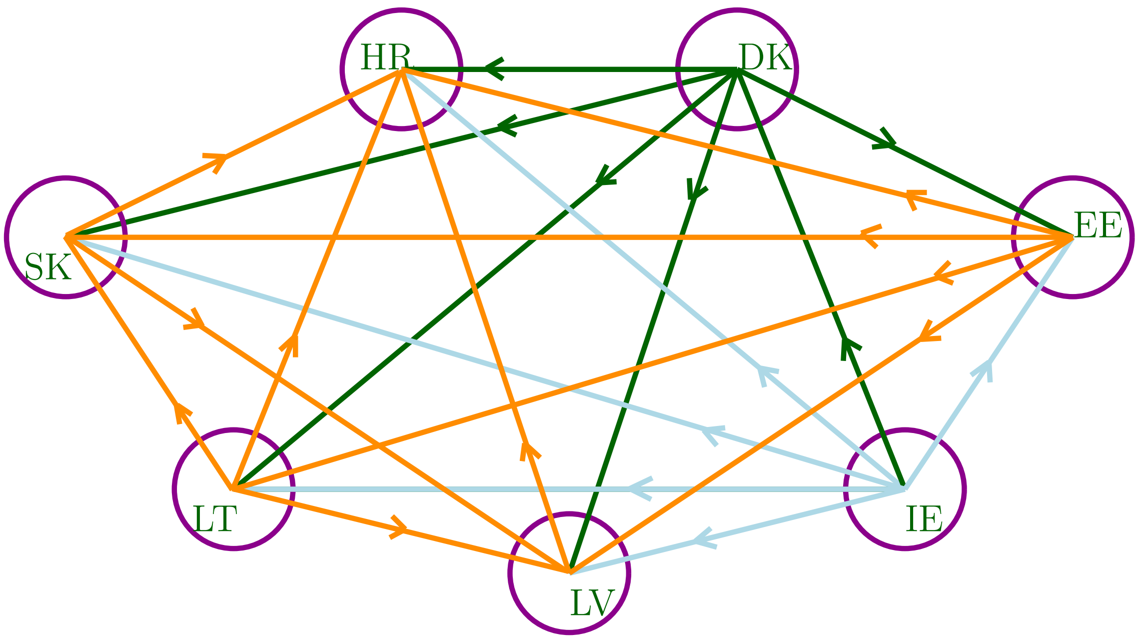

Figure 2: The digraph related to comparing the gross domestic product of selected European Union countries, namely: Croatia (HR), Denmark (DK), Estonia (EE), Ireland (IE), Latvia (LV), Lithuania (LT), and Slovakia (SK)

Although methods based on colouring of graphs that lead to a solution of economic problem are quite rarely used in economics, one can still find some other examples. Beside the utilization in [10], one can find another application in e.g. [9], where the analysis grounded on economic target supported by a special kind of graph colouring (palette colouring) was processed. In this case the stated economic target represent cost saving by organising events in a large exhibition hall, but similar approach to the one presented in [9] can be used in many other situations.

Regarding the cost saving, one can easy find several applications of graph theory - see [7]. At the other side, modelling by graphs is very popular not only in economics, but also in many different areas such as engineering, medicine or biology (see e.g. [19], [10], [28], [29], [30], [36], [37]). Graphs-like structures are very likely used in order to describe the consequences, cause-effect relationships and simulation flows (see e.g. [12], [26], [11]). They have wide applications in logistic especially when dealing with problems of a traffic flow - see e.g. [13], where the study of transport networks in sustainable smart cities is provided. Another example dealing with sustainable development can be find in [3].

The corporate sustainability performance can be studied via graphs too. For instance, the social, economic, environmental, and corporate governance subsets of composite sustainability indicators have been studied in [5]. In the paper authors used models based on the fuzzy similarity graphs and evaluate the resulting five similarity graphs with significantly different topological structures.

The list of applications of graphs in economics is much longer than the one presented above. Mentioned examples provide only a small portion of it.

3 Conclusion

The solution of many economic problems lead themselves naturally to a graph. Some of them had been mentioned above.

In this paper we have shown that graphs are worth not only by reasoning in order to make verbal quantifiers more accurate, and for gathering some economic information in a simple and illustrative way. On a number of examples we have demonstrated their utilization as powerful aids for solving many problems related to different tasks of economy. We have exhibit their usage by classification and prediction, by risk management and cost saving, by modelling and better understanding the variables and their relative interdependencies, by input-output analysis, by studying the consequences and cause-effect relationships. We have shown that graphs are worth for exploring graph-based structural patterns, linkages and their topological properties by which some methods based on colourings of graphs might be of use, as well.

Although the above list of applications is long, it is still far from being complete. This fact underlines the widespread utilization of graphs in economics.

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Bibliography

[1] Alcock, J., Cockcroft, S., Finn, F.: *Quantifying the Advantage of Secondary Mathematics Study for Accounting and Finance Undergraduates.* Accounting & Finance 48(5), (2008), 697–718.

[2] Babichenko, J. S., Babichenko, S. I: *Analysis and Modelling of Work-related Stress Determinants in Cross-Cultural Context.* Transformations in Business & Economics 7(3), (2008), 97–106.

[3] Bates, S., Angeon, V., Ainouche, A.: *The Pentagon of Vulnerability and Resilience: A Methodological Proposal in Development Economics by Using Graph Theory.* Economic Modelling 42, (2014), 445–453.

[4] Bondy, J. A., Murty, U. S. R.: *Graph Theory with Applications.* Vol. 290. London: Macmillan, 1976.

[5] Dočekalová, M. P., Doubravský, K., Dohnal, M., Kocmanová, A.: *Evaluations of corporate sustainability indicators based on fuzzy similarity graphs.* Ecological Indicators 78, (2017), 108–114.

[6] Drabiková, E., Fecková Škrabuľáková, E.: *Decision Trees - A Powerful Tool in Mathematical and Economic Modeling.* In: 18th International Carpathian Control Conference 2017 (ICCC 2017), Danvers, IEEE, 34–39.

[7] Drabiková, E., Fecková Škrabuľáková, E.: *Reducing Costs by Graph Algorithms.* In: 19th International Carpathian Control Conference 2018 (ICCC 2018), Danvers, IEEE, 113–117.

[8] Faisal, M. N., Banwet, D. K., Shankar, R.: *Quantification of Risk Mitigation Environment of Supply Chains Using Graph Theory and Matrix Methods.* European Journal of Industrial Engineering 1(1), (2007), 22–39.

[9] Fecková Škrabuľáková, Grešová, E.: *Costs Saving via Graph Colouring Approach.* Scientific Papers of the University of Pardubice: Series D 45(1), (2019), 152–160.

[10] Fecková Škrabuľáková, Grešová, E., Khouri, S.: *Similarity Assesment of Economic Indicators of Selected Countries by Graph Theory Means.* To appear in Transformations in Business & Economics 47, (2019).

[11] Fedriani, E. M., Tenorio, Á. F.: *Simplifying the Input–Output Analysis through the Use of Topological Graphs.* Economic Modelling, 29(5), (2012)1931-1937.

[12] Flegner, P., Kačur, J., Durdán, M., Laciak, M., Stehlíková, B., Pástor, M.: *Significant Damages of Core Diamond Bits in the Process of Rocks Drilling.* Engineering Failure Analysis 59, (2016), 354–365.

[13] Frutos Bernal, E., Del Rey A. M.: *Study of the Structural and Robustness Characteristics of Madrid Metro Network.* Sustainability 11(12), (2019), 3486.

[14] Haar, J., Schmitz, A., Di Fabio, A., Daellenbach, U.: *The Role of Relationships at Work and Happiness: A Moderated Moderated Mediation Study of New Zealand Managers.* Sustainability 11(12), (2019), 3443.

[15] Halkos, G. E., Tsilika, K. D.: *Assessing Classical Input Output Structures with Trade Networks: A Graph Theory Approach.* MPRA, (2016), 72511.

[16] Herstein, I. N.: *Some Mathematical Methods and Techniques in Economics.* Quarterly of Applied Mathematics 11(3), (1953), 249–262.

[17] Haigh, M. S., Bessler, D. A.: *Causality and Price Discovery: An Application of Directed Acyclic Graphs.* The Journal of Business 77(4), (2004), 1099–1121.

[18] Jacques, I.: *Mathematics for Economics and Business.* Harlow: Pearson Education, 2009.

[19] Kol'veková, G., Liptáková, E., Štrba, Ľ., Kršák, B., Sidor, C., Cehlár, M., Khouri, S., Behún, M.: *Regional Tourism Clustering Based on the Three Ps of the Sustainability Services Marketing Matrix: An Example of Central and Eastern European Countries.* Sustainability 11(2), (2019), 400.

[20] Lautier, D., Raynaud, F.:*Systemic Risk in Energy Derivative Markets: A Graph-theory Analysis.* The Energy Journal 33(3), (2012), 215–239.

[21] Lloret-Climent, M., Nescolarde-Selva, J. A., Alonso-Stenberg, K., Selva-Barthelemy, M. A.: *Causal Analysis of the Spanish Industrial Sector through Smarta.* IEEE Access 7, (2019), 33556–33564.

[22] Meluzin, T., Balcerzak, A. P., Pietrzak, M. B., Zinecker, M., Doubravsky, K.: *The Impact of Rumours Related to Political and Macroeconomic Uncertainty on IPO Success: Evidence from a Qualitative Model.* Transformations in Business & Economics 17(2),(2018), 148–169.

[23] Michaeli E., Solár, V., Vilček, J., Ivanová, M.: *Selected Specific Foods and Agricultural Products with Protected Designation in Regions of the Slovak Republic as a Member State of the European Union.* Journal of International Scientific Publications: Ecology & Safety 12, (2018), 335–340.

[24] Müller, W., Wysotzki, F.: *Automatic Construction of Decision Trees for Classification.* Annals of Operations Research 52(4), (1994), 231–247.

[25] Onnela, J. P., Chakraborti, A., Kaski, K., Kertiész, J.: *Dynamic Asset Trees and Portfolio Analysis.* The European Physical Journal B - Condensed Matter and Complex Systems 30(3), (2002), 285–288.

[26] Pešková, A., Svetlík, J.: *Simulation of Maintenance and Repair of Machines.* Transfer inovácií 31, (2015), 94–96.

[27] Podgorelec, V., Kokol, P., Stiglic, B., Rozman, I.: *Decision Trees: An Overview and Their Use in Medicine.* Journal of Medical Systems 26(5), (2002), 445–463.

[28] Quijada-Alarcón, J., González Cancelas, N., Soler Flores, F. J., Camarero Orive, A.:*Use of Decision Trees Algorithm for the Territorial Logistic Planning.* Revista Pensamiento Matemático 3(2), (2013), 51–58.

[29] Rao, R. V.: *A Material Selection Model Using Graph Theory and Matrix Approach.* Materials Science and Engineering: A 431(1-2),(2006), 248–255.

[30] Ráškayová, D., Laciak, M., Flegner, P., Kačur, J., Durdán, M.: *Optimization of Control Parameters of Production Refractory Materials in Tunnel Furnace.* In: 19th International Carpathian Control Conference 2018 (ICCC 2018), Danvers, IEEE, 104–107.

[31] Rokach, L., Maimon, O. Z.: *Data Mining with Decision Trees: Theory and Applications.* Vol. 69., Danvers: World scientific, MA, USA, 2008.

[32] Sandhu, R. S., Georgiou, T. T., Tannenbaum, A. R.*Ricci Curvature: An Economic Indicator for Market Fragility and Systemic Risk.* Science Advances 2(5), (2016), 1–10.

[33] Škrabuľáková, E., Marasová, D., Tomášková, M., Grendel, P.: *Identification of Factors Causing Damage to Conveyor Belts during the Transportation of Coal.* Applied Mechanics and Materials 683, (2014), Trans Tech Publications, Switzerland, 130–134.

[34] Torres, J. M., Duenas-Osorio, L., Li, Q., Yazdani, A.*Exploring Topological Effects on Water Distribution System Performance Using Graph Theory and Statistical Models.* Journal of Water Resources Planning and Management 143(1), (2016), 04016068.

[35] Vitali, S., Glattfelder, J.B., Battiston, S.: *The Network of Global Corporate Control*. PLoS ONE 6(10), (2011), e25995.

[36] Wagner, S. M., Neshat, N.: *Assessing the Vulnerability of Supply Chains Using Graph Theory*. International Journal of Production Economics 126(1), (2010), 121–129.

[37] Zdimalova, M., Roznovjak, R., Weismann, P., El Falougy, H., Kubikova, E.: *Use of Graph Algorithms in the Processing and Analysis of Images with Focuse on the Biomedical Data*. Bratislava Medical Journal 118(8), (2017), 485–490.

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Study of mathematics and economics: View from the other side

Elena Grešová[1] **and Erika Fecková Škrabuľáková**[2]

Abstract: In this paper we were interested in the opinion of students of non-mathematical, non-economic fields of study on mathematics and economics and the usage of knowledge from these disciplines in practice. We seek to find the connection between the practice and teaching of mathematics and economics. We consider the research results being important indicators of the teaching process reflection. The aim is to comprehend the students' point of view and provide feedback. In order to determine students' beliefs a questionnaire technique was applied. The key claims were formed on the basis of analyzing the gathered data and their further matching. In order to better process the obtained information, all the data were normalized. The grounds of uncovered findings were examined, too. This paper presents a view on contemporary teaching and offers suggestions for making improvements in the studied area.

Keywords: Economics; Education; Mathematics; Student's perspective.

Mathematics Subject Classification: 97C70, 97A40, 97B40.

1 Introduction

Most of the universities in Europe arrange mathematics as one of the compulsory subjects that students need to pass during the study, no matter what their field of study is. Thus, mathematics unnecessarily becomes a necessary turning point in the path to successful graduation. The required status of mathematics

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evokes negative emotions towards it and distorts, its real need in practice. Therefore, we were interested in the opinion of students of non-mathematical fields of study on mathematics and the usage of knowledge from it in practice. As the fundamentals of economics are taught at many universities in many different forms for students of non-economic disciplines, we were also interested in the students' view on the study of these subjects and the use of acquired knowledge from economics in practice. In order to get this information, we contacted the students with a short questionnaire. In this article, we analyze the information obtained from in total 150 students of Technical University of Košice and confront the results in a subject-matter context.

2 Literature overview

Research in the field of linking teaching with practice was carried out by Donckels [4]. The central idea was to support entrepreneurial talents already in the educational process. The author emphasized the need of university's proactive approach in developing student's potential, which is a direct benefit to the country's economic growth. A survey discussing education and business links was accomplished. What is very significant, attention was paid not only to the attitude of government institutions, businesses or schools on this issue, but also to the attitudes of the students themselves. The resulting benefits of the analysis of the survey results are proposals for concrete solutions.

Students view on mathematics teaching at university has been the subject of research interest of Dalby et al. [3]. The results showed that students would appreciate a greater degree of synthesis between school mathematical tasks and practice. They require more detailed information on the mathematical content of education and better orientation in the areas of practical mathematics usage. About 32 % of respondents held these views. Approximately every third student asked for more use of so-called "linking maths" - connecting mathematics that shows ways of solving problems while selecting examples from practice.

Further research of Flegg, Mallet and Lupton [5] focused on the students' attitudes to linking their study with practice. Authors concentrated on technical disciplines. More precisely, they focused on engineering fields which they see as an application of science and mathematics to design and build diverse projects serving society. Authors followed the students' opinions on the use of various mathematical techniques for a wide range of application alternatives (see also [10]). In addition to using the acquired information to successfully master mathematics as one of the school subjects that they have to complete, or using the above mentioned techniques as a subsidiary tool in manifold technical subjects, it was mainly possible to tackle real situations and problems. Several improvements

have been identified that have been designed to be implemented in the curriculum. They were related to the overall significance of mathematics in the context of engineering disciplines, future studies, or the effectiveness of solving practical tasks.
As evidenced by Jacques's book [6] mathematical knowledge is necessary in order to acquire financial literacy and general knowledge of economics. Jacques's book belongs to the many works aimed at gaining and improving the mathematical basis for students of economics, business or management. A clear correlation was found between the results of successful completion of economic subjects at universities and the knowledge of mathematics [2]. The importance of mathematics for the study of economists was confirmed by research of Alcock, Cockcroft and Finn [1]. Their study suggests that students who have completed more advanced mathematical subjects showed a significantly higher success in economics courses with practical focus.

An interesting fact is that students tend to overestimate their knowledge of mathematics. The results of studies by Pajares and Miller (see [7] and [8]) have shown that up to 86 % of high school students and 60 % of undergraduates tend to overestimate their mathematical knowledge. Only one in twenty high school students and one in five university students realistically evaluate their mathematics ability. The overly positive perception of mathematical preparedness by students on the basis of their self-assessment was also demonstrated by Wandel et al. [9].

3 Results and discussion

In this paper we present the results of a questionnaire of in total 150 students of Technical University in Košice, Košice, Slovakia, whose major subject is neither mathematics, nor economics or management, but for whom these subjects appeared during their study in different forms in several university or high school level courses. In the questionaire we asked in total 6 questions $\{Q_1, Q_2, \ldots, Q_6\}$, namely, whether mathematics/economics is useful in practice, what is it useful for and whether the students encountered mathematics/economics at all. As students were encouraged to create their answers themselves, they were allowed to give more than one answer to each question. Therefore, in order to better compare the data, all the obtained answers were grouped into several sets according to their similarity and we counted the number of answers associated with each set s_i and the total number of answers for each question q_j; $j \in \{1, 2, \ldots, 6\}$. The data were normalized via formula:

$$p_{ij} = \frac{s_i}{q_j} \cdot 100.$$

Thus, we obtained the normalized values p_{ij} of answers to q_j^{th} question Q_j belonging to i^{th} set indicated in percentage.

Our research shows that the respondents of the Technical University of Košice, Košice, Slovakia, perceive to the same extent the personal need for economics from the perspective of the development of cognitive functions and from the perspective of solving various situations of everyday life (see Figure 1). Respondents largely understand the necessity for knowledge of economics for use in their employment. They do not perceive this need only in the context of passing school. Likewise, no respondent said he did not know what economics was for.

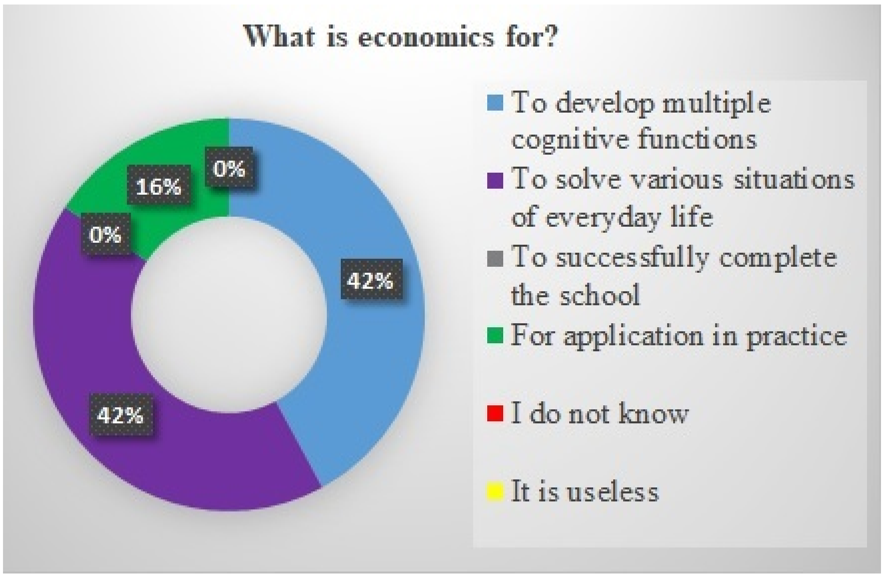

Figure 1: What is economics for?

The interviewed students perceived a personal need for mathematics, especially from the perspective of development of different cognitive functions - see Figure 2. Roughly one third was aware of a need for mathematics in order to deal with divergent situations of everyday life, of which every second saw these situations from a financial perspective. Every fifth student perceived mathematics as a subject needed to successfully complete their schooling. Up to 6 % of respondents were unable to find the answer to the question “What is mathematics for?”. What is sad, 2 % of respondents even said that it is useless. In contrast, only 3 % recognized its real application in practice.

Also interesting is a comparison of the answers to the question “What is mathematics for?” between the students of first and second degree of university study. Bachelor degree students, Group B, understand that mathematics is for solving manifold situations of everyday life (see the left-down part of Figure 2). They argue with daily situations in stores where they usually make some simple calculations, hence, they see the utilization of basic mathematics. The application of higher mathematics they interconnect more or less only with the development of multiple cognitive functions. A large amount of bachelor degree students see mathematics only as a compulsory subject on the way to successful termination of the university. No one of the Group B students answered that mathematics is useful for application in practice. At the other side, in total 8 % of the students of this group answered that they have no idea what they need mathematics for or, that mathematics is useless. A similar situation occurred in the Group I (engineer or master degree students), where 8 % of students did not know what is mathematics for. On the other hand, we can observe a decreased number of answers in the set of successful termination of school, and in increased number of answers in the set of application in work (see the right-down part of Figure 2). In the Group I the highest percentage of answers was associated with the set of development of cognitive functions.

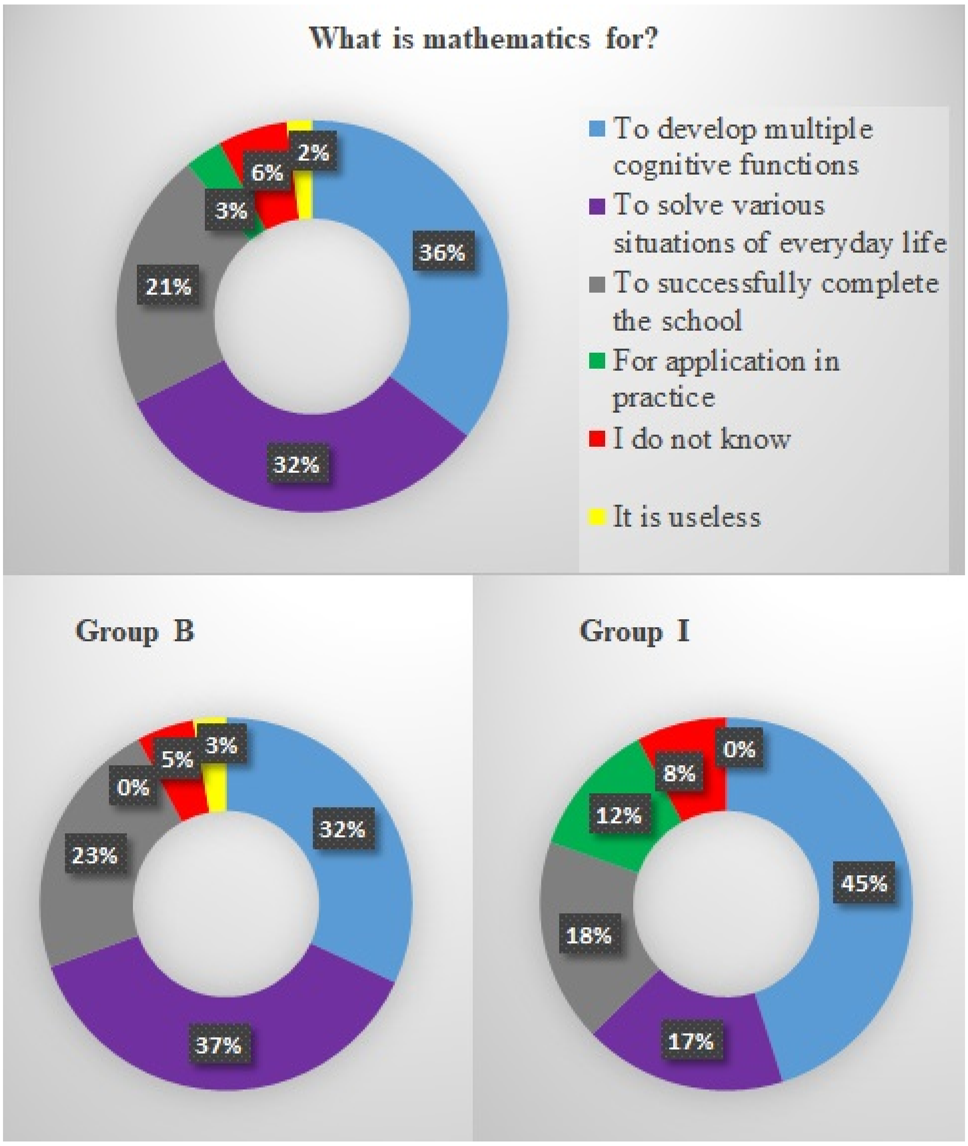

Figure 2: What is mathematics for?

Where do I meet...	**mathematics?** [%]	**economics?** [%]
Everywhere / Almost everywhere in everyday life	58.47	54.67
At university	30.26	0.00
At work	9.74	40.00
Hardly anywhere	0.51	5.33
I do not know	0.51	0.00
It is useless	0.51	0.00

Table 1: Where do I meet mathematics/economics?

The most significant share of answers to the question "Where do I encounter mathematics?" respectively "Where do I meet economics?" has the answer "Everywhere / almost everywhere in everyday life." (see Table 1). The second most significant share in the case of mathematics was the answer: "At school." It is interesting to note that no student in the case of economics reported this option. On the other hand, the second most presented answer to the question "Where do I meet economics?" was the answer "At work." This occurred in up to 40 % of cases. Only less than 10 % of students recorded this answer in the case of mathematics. Since none of the students indicated that economics is inapplicable in practice, respectively does not know where it is usable, and only a few students have indicated such an answer in the case of mathematics, it can be stated that students are aware of the interconnection of these subjects with real life situations. Students approximately equally perceived meeting with mathematics and economics in everyday life, while the first one was often perceived financially, so through economically related aspect.

On the question "Is economics useful in practice?" 85 % of respondents gave a clear positive response. The remaining 15 % saw its usability as conditioned by the type of activity performed, the type of employment - see Figure 3.

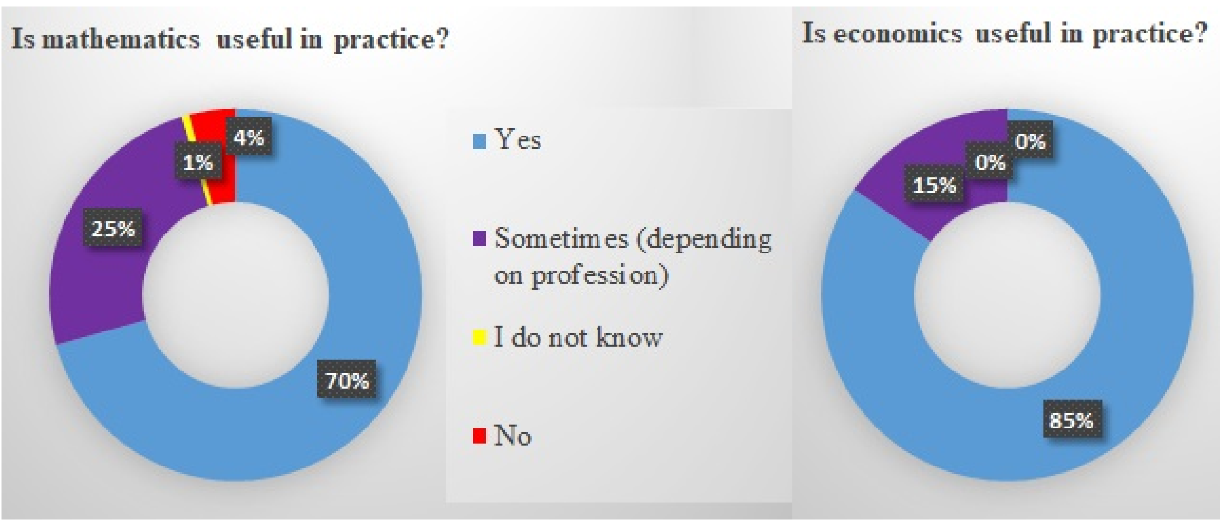

Figure 3: Is mathematics/economics useful in practice?

Although 70 % of students considered mathematics useful in practice and 25 % saw it depending on the profession, there were still 4 % of students that considered it useless. This is very alarming, as some of these students learn the mathematics a couple of semesters at their university. Despite that, they do not see its practical usage.

4 Conclusion

In this paper we were interested in the opinion of students of non-mathematical, non-economic fields of study on mathematics and economics and the usage of knowledge from them in practice. We consider the research results to be important indicators of the teaching process reflection. They provide feedback, a student perspective on the issue of educational curriculum.

The positive result was that many students were aware of the need for mathematical and economic education not only for the development of memory, thinking and other cognitive functions, but also for its use in solving various situations in life. Undergraduate students did not perceive mathematics as a discipline necessary for their own practice. At the other side, in the case of economics they could see the connection between the knowledge obtained in the educational process and its use in work and daily life. Yes, they saw this connection in the case of mathematics as well, but to a lesser extent and often only in financial terms. Very often they limit the usage of mathematics only to the school environment.

The questions of general applicability of mathematics and economics in practice were in most cases answered positively, although the difference between the students' perception of the need for mathematical and economic education was shifted to the detriment of the former. This points out the need for incorporation of subjects that use mathematical knowledge in solving practical tasks to the university students' curriculum. The examples of practical use of mathematics should not be based solely on the financial literacy elements, or use in physics. They need to be demonstrated on a diverse spectrum of real situations. If we want to make students see the meaning of acquiring specific knowledge of mathematics, even of the higher one, we need to develop not only knowledge, but the practical abilities they see in the real world, as well.

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Bibliography

[1] Alcock, J., Cockcroft, S., Finn, F.: *Quantifying the Advantage of Secondary Mathematics Study for Accounting and Finance Undergraduates.* Accounting & Finance 48(5), (2008), 697–718.

[2] Ballard, C. L., Johnson, M. F.: *Basic Math Skills and Performance in an Introductory Economics Class.* The Journal of Economic Education 35(1), (2004), 3–23.

[3] Dalby, T., Robinson, C., Abdulla, S., Galligan, L., Frederiks, A., Pigozzo, R., Wandel, A.: *Students' Mathematical Preparation Part B: Students' Perceptions.* In: Proceedings of the 9th Delta Conference of Teaching and Learning of Undergraduate Mathematics and Statistics 2013, University of Western Sydney, 2013.

[4] Donckels, R.: *Education and Entrepreneurship Experiences from Secondary and University Education in Belgium.* Journal of Small Business & Entrepreneurship 9(1), (1991), 35–42.

[5] Flegg, J., Mallet, D., Lupton, M.: *Students' Perceptions of the Relevance of Mathematics in Engineering.* International Journal of Mathematical Education in Science and Technology 43(6), (2012), 717–732.

[6] Jacques, I.: *Mathematics for Economics and Business.* Harlow: Pearson Education, 2009.

[7] Pajares, F.: *Self-efficacy Beliefs and Mathematical Problem-solving of Gifted Students.* Contemporary Educational Psychology 21(4), (1996), 325–344.

[8] Pajares, F., Miller, M. D.: *Role of Self-efficacy and Self-concept Beliefs in Mathematical Problem Solving: A Path Analysis.* Journal of Educational Psychology 86(2), (1994), 193–203.

[9] Wandel, A. P., Robinson, C., Abdulla, S., Dalby, T., Frederiks, A., Galligan, L. : *Students' Mathematical Preparation: Differences in Staff and Student Perceptions.* International Journal of Innovation in Science and Mathematics Education 23(1), (2015), 82–93.

[10] Ždímalová, M.: *Netradičný prístup k vyučovaniu matematiky.* (*Non-traditional approach to teaching mathematics.*) In: PYTAGORAS 2004. Zvolen: Jednota Slovenských matematikov a fyzikov (2004), 125–129. (In Slovak)

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Entropy concept: A mathematical revision

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Abstract: In this paper, we deal with the commonly used concept of entropy from a mathematical point of view. We revise all the important and known facts about Shannon-Khinchin axioms and the formulation of the classical Shannon entropy, as well as its one-parameter generalizations with the corresponding axioms. We provide an improved and proper derivation of the Shannon entropy formula in more details with a clarification of using the axioms. Moreover, we describe some real-life figurative applications from physics and the information theory with an explanation for better understanding the concept.

Keywords: entropy; Shannon-Khinchin axioms; one-parameter generalization.

Mathematics Subject Classification: 28D20, 94A15, 94A17.

1 Introduction

The concept of entropy was developed between 1872 and 1875. It originated in physics and was first mathematically described by Ludwig Boltzmann in thermodynamics. Then Josiah Gibbs improved this formula in order to be more precise in describing real phenomena. Later, with the development of the information theory, there was a need to mathematically define entropy more rigorously. In 1948, Claude Shannon in [5] introduced a revised entropy for applications in this field and he also postulated axioms, which should uniquely describe this entropy. However, the correct and precise formulation of the axioms was given after nine years in [4] by Aleksandr Khinchin.

Even though entropy is now known and studied for a long time, it can be considered as the basic concept for some new and innovative approaches in the

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probability theory and informatics. Many algorithms and methods are based on the original entropy and its generalizations.

Main contribution of this paper, besides organized revision of the known facts in the area, is thorough understanding of the mathematical background as well as the whole concept of entropy. That is why we give some examples from physics and the information theory so that its interpretation is clearer and we also highly focus on a derivation of the entropy formula, where we include all the details and necessary tools.

2 Interpretation

2.1 Physics

In physics, entropy appears in statistical physics and describes the degree of disorder in a system [2]. It links together microscopic and macroscopic characteristics of it in a way that entropy is proportional to the number of all possible microscopic configurations (orderings) of molecules in the system while an external observer does not notice any difference of the overall macroscopic result. Therefore entropy is connected with uncertainty in the molecular configuration.

According to the third law of thermodynamics, entropy of a closed system never decreases, so it is constant (when reaching equilibrium) or increasing. That is why we can determine the direction of all processes in nature. So that only knowing the entropy values for two different states of some system, we can derive their order.

Consider an arbitrary room in an arbitrary house. The longer we use it, the messier it gets, so that its entropy is increasing. However, when we decide to clean the room, does the entropy decrease? The right answer is no. Even though it is less messy, the person cleaning is tired and sweaty, so his entropy has increased. It means that the entropy of the entire closed system which contains the whole room as well as the person has not really decreased.

2.2 Information theory

In the information theory, entropy is connected with the degree of uncertainty or the amount of freedom one has choosing parameters in creation of a message, see [5].

Consider a system which consists of letters

$$A\ B\ C\ D\ E\ F\ G\ H\ I\ J\ K\ L\ M\ N\ O\ P\ R\ S\ T\ U\ V\ W\ X\ Y\ Z.$$

When someone chooses a letter from this system, how many questions (with answer yes or no) do you minimally need to find out, what is the chosen one? You can start guessing the letter randomly or from the beginning, but there are more efficient ways. One of them is called binary search and is based on dividing the system into two subsystems, then the right subsystem into two sub-subsystems and so on. So in this case, you will need up to five questions.

Now assume another system containing

$$A\ A\ A\ A\ A\ A\ A\ A\ A\ A\ A\ A\ A\ A\ A\ A\ A\ A\ A\ A\ A\ A\ A\ A\ A\ A,$$

which has as many letters as the first one. How many yes-no questions do you need now? Apparently, you know the answer even without asking.

As we see according to the examples, entropy is proportional to the number of questions. The more questions are needed, the greater information the system contains, the higher the uncertainty is and the higher the entropy is.

3 Axioms

Before presenting the axioms, there is a necessity to understand some basic ideas in the probability theory. Special types of probability used in this paper are joint (marked p_{ij}), which describes probability of two (or more) different events occurring together/at the same time and marginal (marked $p_{i\bullet}$), which describes probability of a single event occurring unconditioned on any other. The following table shows the idea of creating marginal probabilities from the joint ones and then we also write relations between them in a mathematical way.

p_{11}	p_{12}	p_{13}	$\cdots$	p_{1m_1}	$\rightarrow$	$p_{1\bullet}$
p_{21}	p_{22}	p_{23}	$\cdots$	p_{2m_2}	$\rightarrow$	$p_{2\bullet}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
p_{n1}	p_{n2}	p_{n3}	$\cdots$	p_{nm_n}	$\rightarrow$	$p_{n\bullet}$

For any $i = 1, \ldots, n$ and $j = 1, \ldots, m_i$ it is satisfied, that

$$p_{ij} \geq 0, \qquad p_{i\bullet} = \sum_{j=1}^{m_i} p_{ij}, \qquad \sum_{i=1}^{n} p_{i\bullet} = 1, \tag{1}$$

where in each row of the table, there is at least one non-zero element, so

$$(\forall i \in \{1, \ldots, n\})\ (\exists j \in \{1, \ldots, m_i\})\ p_{ij} > 0,$$

and it means that all marginal probabilities are always positive ($p_{i\bullet} > 0$).

As was mentioned in the Introduction, the proper way of defining Shannon entropy is given by Shannon-Khinchin axioms. These axioms can be separated into two different groups. The first one is natural, mainly derived from physics, because of real properties of the physical quantity they describe (maximality and weaker formulation of additivity). The second one is mathematical, because of the uniqueness of the entropy formulation and its simplicity when using in calculations (continuity and expandability).

Here, we consider Shannon-Khinchin axioms according to [7], where

$$\Delta_n = \left\{ (p_1, \ldots, p_n) : (\forall i)\, p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}. \tag{2}$$

There exists only one class of functions S_n (n is related to number of elements in the argument), which satisfies following properties:

SK1 *Continuity*
For any $n \in \mathbb{N}$, function $S_n(p_1, \ldots, p_n)$ is continuous with respect to $(p_1, \ldots, p_n) \in \Delta_n$.

SK2 *Maximality*
For given $n \in \mathbb{N}$ and $(p_1, \ldots, p_n) \in \Delta_n$, function $S_n(p_1, \ldots, p_n)$ takes its largest value for $p_i = \frac{1}{n}$ $(i = 1, 2, \ldots, n)$

$$S_n(p_1, \ldots, p_n) \leq S_n\left(\frac{1}{n}, \ldots, \frac{1}{n}\right).$$

SK3 *Shannon additivity*
Using marginal and joint probabilities as in (1) and $(p_1, \ldots, p_n) \in \Delta_n$, following equality holds

$$S_{\sum_i m_i}(p_{11}, \ldots, p_{nm_n}) = S_n(p_1, \ldots, p_n) + \sum_{i=1}^{n} p_i S_{m_i}\left(\frac{p_{i1}}{p_{i\bullet}}, \ldots, \frac{p_{im_i}}{p_{i\bullet}}\right).$$

SK4 *Expandability*
Assuming $(p_1, \ldots, p_n) \in \Delta_n$, adding an event with zero probability does not change the value

$$S_{n+1}(p_1, \ldots, p_n, 0) = S_n(p_1, \ldots, p_n).$$

According to these four axioms, the unique formulation of the entropy is given as

$$S_n(p_1, \ldots, p_n) = -k \sum_{i=1}^{n} p_i \ln p_i,$$

and is called *Shannon entropy* [5], where k is a positive constant. In physics k is called *Boltzmann constant* with a value $k = 1{,}38 \cdot 10^{-23}\,\mathrm{m}^2\,\mathrm{kg}\,\mathrm{s}^{-2}\,\mathrm{K}^{-1}$. In the information theory $k = 1$, because it is just a reference value. We also need to use a convention for the case where p_i equals zero, so that $0 \cdot \log 0 = 0$.

From the formulation, some properties of the entropy can be derived, such as symmetry of its arguments (any permutation of probabilities will not change the value), concavity, non-negativity and its equality to zero if and only if there exists an event with probability one.

Now, we will give a derivation of the entropy formula, according to [5]. Let us define $S_u\left(\frac{1}{u},\frac{1}{u},\dots,\frac{1}{u}\right)=A(u)$ as the entropy of u events with the same probability. If we consider b^c independent events, using the additivity axiom (SK3) we can write

$$\begin{aligned} A(b^c) &= S_{b^c}\left(\frac{1}{b^c},\dots,\frac{1}{b^c}\right) = S_b\left(\frac{1}{b},\dots,\frac{1}{b}\right)+\sum_{i=1}^{b}\frac{1}{b}S_{b^{c-1}}\left(\frac{1}{b^{c-1}},\dots,\frac{1}{b^{c-1}}\right)\\ &= A(b)+A(b^{c-1}). \end{aligned}$$

By induction we obtain equation

$$A(b^c)=c\,A(b). \tag{3}$$

Consider[2] $t,n,s\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$. Now we will take t^n events. For every n we can find m, such that for every s, $s\geq 2$ it is true that

$$s^m\leq t^n<s^{m+1}. \tag{4}$$

Taking the logarithm of (4) and by additional changes we get

$$\frac{m}{n}\leq\frac{\log t}{\log s}\leq\frac{m}{n}+\frac{1}{n},\quad \text{i.e.,}\quad \left|\frac{m}{n}-\frac{\log t}{\log s}\right|<\frac{1}{n}. \tag{5}$$

Moreover, from the maximality (SK2) and the expandability axiom (SK4) applied to (4) we obtain

$$A(s^m)\leq A(t^n)\leq A(s^{m+1}).$$

Expanding as in (3) we get $A(s)\leq nA(t)\leq(m+1)A(s)$ and, by the same technique as above, we can write

$$\frac{m}{n}\leq\frac{A(t)}{A(s)}\leq\frac{m}{n}+\frac{1}{n},\quad \text{i.e.,}\quad \left|\frac{m}{n}-\frac{A(t)}{A(s)}\right|<\frac{1}{n}. \tag{6}$$

Using together both results (5) and (6), we obtain

$$\left|\frac{A(t)}{A(s)}-\frac{\log t}{\log s}\right|\leq\frac{2}{n}.$$

Since n is arbitrarily big, it follows

$$\frac{A(t)}{\log t}=\frac{A(s)}{\log s},$$

[2]entropy is associated with big numbers (representing molecules or bits of information) so we do not need to assume small ones

and from this formula it is obvious that $A(t)/\log t$ is a constant function and

$$A(t) = k \log t, \tag{7}$$

where k is a positive constant.

Assuming the additivity axiom (SK3), we use (7) for the equiprobability distribution of considered events on the left hand side, where
$p_{11} = p_{12} = \cdots = p_{nm_n} = \left(\sum_{j=1}^n m_j\right)^{-1}$,

$$S_{\sum_i m_i}(p_{11}, \ldots, p_{nm_n}) = A\left(\sum_{j=1}^n m_j\right) = k \log\left(\sum_{j=1}^n m_j\right),$$

as well as in the second expression on the right hand side, where $\frac{p_{im_i}}{p_{i\bullet}} = \frac{1}{m_i}$, because $p_{i1} = p_{i2} = \cdots = p_{im_i}$

$$S_{m_i}\left(\frac{p_{i1}}{p_{i\bullet}}, \ldots, \frac{p_{im_i}}{p_{i\bullet}}\right) = A(m_i) = k \log m_i.$$

Replacing them in the original axiom we get

$$k \log\left(\sum_{j=1}^n m_j\right) = S_n(p_1, p_2, \ldots, p_n) + k \sum_{i=1}^n p_i \log m_i.$$

Now, using $\sum_{i=1}^n p_i = 1$ and by some equivalent changes, the entropy equals

$$S_n = k\left[\sum_{i=1}^n p_i \log\left(\sum_{j=1}^n m_j\right) - \sum_{i=1}^n p_i \log m_i\right] = k \sum_{i=1}^n p_i \log\left(\frac{\sum_{j=1}^n m_j}{m_i}\right).$$

Thanks to the continuity axiom (SK1) we can describe probability of the i-th event by formula $p_i = \frac{m_i}{\sum_{j=1}^n m_j}$ and get the right formulation of the entropy

$$S_n = k \sum_{i=1}^n p_i \log\left(\frac{1}{p_i}\right) = -k \sum_{i=1}^n p_i \log p_i. \qquad \square$$

4 Other types of entropy

Because of the progress in science there was a need to define new types of entropy. They should have the same basic properties as Shannon's and moreover better describe real phenomena. The most common and used entropies of this type are Rényi entropy [1] (used e. g. in ecology and cryptography) and Tsallis entropy [3], [6] (used e. g. in image processing and plasma physics). They both are one-parameter generalizations of Shannon entropy.

4.1 Rényi entropy

If we consider Δ_n as in (2), Rényi entropy is defined as

$$S_q^R(p_1, \ldots, p_n) = \frac{k}{1-q} \ln \left(\sum_{i=1}^{n} p_i^q \right).$$

It can be easily shown that, in the limit $q \to 1$, it reduces to Shannon entropy (using L'Hospital rule), i.e.,

$$\lim_{q\to 1} S_q^R = k \lim_{q\to 1} \left[\frac{\ln \left(\sum_{i=1}^{n} p_i^q \right)}{1-q} \right] = k \lim_{q\to 1} \frac{\sum_{i=1}^{n} p_i^q \ln p_i}{-\sum_{i=1}^{n} p_i^q} = -k \sum_{i=1}^{n} p_i \ln p_i = S_n.$$

The same maximality property as for Shannon entropy also holds, so the maximum is attained when all the probabilities are equal to $p_i = \frac{1}{n}$. They also share the continuity and the expandability property. Moreover, it is easily seen that this entropy is always non-negative (minus sign and a negative logarithm of the value in $(0,1)$).

4.2 Tsallis entropy

Using Δ_n as in (2), definition of Tsallis entropy reads as

$$S_q^T(p_1, \ldots, p_n) = \frac{k}{q-1} \left(1 - \sum_{i=1}^{n} p_i^q \right).$$

Since Tsallis and Rényi entropy have very similar formulations, we can find a connection between them in the form

$$S_q^T = \frac{k}{q-1} \left[1 - \exp\left(\frac{(q-1) S_q^R}{k} \right) \right].$$

Because of this equation, it is evident that Tsallis entropy is a monotone function of Rényi entropy and any maximum of Tsallis entropy will also be a maximum of Rényi entropy, and vice versa. Moreover, it reaches its maximum for the equiprobability distribution.

Tsallis entropy tends to Shannon entropy in the limit $q \to 1$, since

$$\lim_{q\to 1} S_q^T = k \lim_{q\to 1} \left[\frac{1 - \sum_{i=1}^{n} p_i^q}{q-1} \right] = k \lim_{q\to 1} \frac{-\sum_{i=1}^{n} q\, p_i^q \ln p_i}{1} = -k \sum_{i=1}^{n} p_i \ln p_i = S_n.$$

Non-negativity of the entropy can be easily verified (value in the brackets is always non-negative).

4.3 q-generalized entropy

From a mathematical point of view, it is also interesting to look at one-parameter entropies in general. The main reason is that they have similar characteristics and also satisfy three out of four Shannon-Khinchin axioms. The only difficulty is with the additivity, so there should be a new formulation for this property. Moreover, this generalization should tend to the original Shannon additivity in the limit.

Such a generalization is called q-generalization and it contains axioms for a function $\phi(q)$ as well as axioms for the whole entropy, which are similar to Shannon-Khinchin axioms [7], [8]. Using them, we can uniquely determine one-parameter generalization of Shannon entropy as

$$S_q^G(p_1, \ldots, p_n) = \frac{1 - \sum\limits_{i=1}^{n} p_i^q}{\phi(q)}$$

with $q \in \mathbb{R}^+$, where $\phi(q)$ satisfies properties (i) – (iv):

(i) $\operatorname{sgn} \phi(q) = \begin{cases} +1, & \text{for } q \geq 1, \\ -1, & \text{for } q \in (0,1), \end{cases}$

(ii) $\phi(q)$ is differentiable with respect to q,

(iii) $\lim\limits_{q \to 1} \frac{\mathrm{d}\phi(q)}{\mathrm{d}q} = 1$,

(iv) $\lim\limits_{q \to 1} \phi(q) = \phi(1) = 0$ with $\phi(q) \neq 0$ fo $q \neq 1$.

These axioms guarantee that the formulation meets the original Shannon entropy in the limit $q \to 1$, which can be seen applying L'Hospital rule.

The q-generalized axioms consist of four conditions (for simplification we will not write number of elements in the argument of the entropy):

qSK1 *Continuity*

S_q^G is continuous in Δ_n.

qSK2 *Maximality*

For any $n \in \mathbb{N}$ and any $(p_1, \ldots, p_n) \in \Delta_n$

$$S_q^G(p_1, \ldots, p_n) \leq S_q^G\left(\frac{1}{n}, \ldots, \frac{1}{n}\right).$$

qSK3 *Generalized Shannon additivity*

Using marginal and joint probability from (1) and $(p_1, \ldots, p_n) \in \Delta_n$, following holds

$$S_q^G(p_{11}, \ldots, p_{nm_n}) = S_q^G(p_1, \ldots, p_n) + \sum_{i=1}^{n} p_i^q S_q^G\left(\frac{p_{i1}}{p_i}, \ldots, \frac{p_{im_i}}{p_i}\right).$$

qSK4 *Expandability*

For $(p_1, \ldots, p_n) \in \Delta_n$

$$S_q^G(p_1, \ldots, p_n, 0) = S_q^G(p_1, \ldots, p_n).$$

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Bibliography

[1] P. A. Bromiley; N. A. Thacker; E. Bouhova-Thacker: *Shannon entropy, Rényi entropy, and information.* Stat. Inf. Series, (2000), 2004-004.

[2] C. G. Chakrabarti; K. De: *Boltzmann–Gibbs entropy: Axiomatic characterization and application.* Int. J. Math. Math. Sci. 23(4), (2000), 243–251.

[3] S. Furuichi: *On uniqueness theorems for Tsallis entropy and Tsallis relative entropy.* IEEE Trans. Inf. Theory 51(10), (2005), 3638-3645.

[4] A. Y. Khinchin: *Mathematical foundations of information theory.* Dover Publications, Inc., New York, (1957), translated.

[5] C. E. Shannon: *A mathematical theory of communication.* Bell Syst. Tech. J. 27(3), (1948), 379–423.

[6] A. Sumiyoshi: *Axioms and uniqueness theorem for Tsallis entropy.* Phys. Letters A 271(1-2), (2000), 74–79.

[7] H. Suyari: *Generalization of Shannon–Khinchin axioms to nonextensive systems and the uniqueness theorem for the nonextensive entropy.* IEEE Trans. Inf. Theory 50(8), (2004), 1783–1787.

[8] T. Wada; H. Suyari: *A two-parameter generalization of Shannon–Khinchin axioms and the uniqueness theorem.* Phys. Letters A 368(3-4), (2007), 199–205.

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A method to identify states of the experimental phases for the process of the transport belt tensioning and relaxation

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Abstract: In this paper we present a method which was applied in order to identify states of the particular experimental phases for the process of the transport belt tensioning and relaxation. The test equipment of the hexagonal idler housing of the pipe conveyor emitted tension force of conveyor belt to the size of contact forces which are induced by the balled conveyor belt on the guide rollers. Using our method to identify the state tension and state relaxation enables us to reduce useless data variability. We discuss the reliability of the results obtained via experimental approach compared to the mathematical - theoretical one. The described method is based on the differences of tension force time behaviors. The method is applied during the processing of data after the measurement before estimate of relations, functions, rules between tension forces and contact forces. All measurement results processed by this method can be used in the future to analyze the significance of factors, model parameter estimates, and more.

Keywords: Pipe conveyor; Relaxation phase; Tension phase; Data processing.

Mathematics Subject Classification: 62K99, 74-05.

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1 Introduction

Mathematics plays a significant role in various phases of engineering research, see e.g. [4]. Mathematical tools help to derive formulas and relations that describe the behaving of real objects. The question is the extent to which these formulas correspond to the reality. In this paper we aim to use the experiment in order to identify the state of experimental phases for the process of the transport belt tensioning and relaxation and discuss the reliability of the results obtained by the experiment.

Belt conveyor is among the most important tools for the transportation of bulk material [12]. Pipe conveyors are considered to be the most perspective system for transportation of bulk and lump materials, but various factors might have an impact on their functionality - see e.g. [11]. The determination of forces in the conveyor belt is very complicated and the currently used procedures are inaccurate because the conveyor belt is a hyperplastic material with orthotropic behavior [5]. The forces are therefore determined by experimental measurements. At the Technical University of Košice we designed the test equipment of the hexagonal idler housing of pipe conveyor. The concept of it represents the section of pipe conveyor in which the conveyor belt is transformed to the pipe form. The conveyor belt as a physical model is an object with action of tension force and at the same time it takes the initiative of the contact force formation.

Although the theoretically deduced formulas describe the situation at various positions of the pipe conveyor, experimental results from measurements on the device can bring better insight into the actual distribution of forces in the pipe conveyor.

2 Literature overview

The scope of the design and development of the belt conveyor control system is to minimize the necessary amount of physical labor, to reduce energy consumption, and to reduce the amount of information required, and thus to overall increase the global efficiency of the conveyor operation and reduce the occurrence of accidents [3]. The experiments will allow the calculation and prediction of the conveyor belt service life under the specific operating conditions. A test rig was constructed by Chen et al. [10] and Petrikova [9].

The major purpose of the data analysis is to explicate, interpret, and describe the process using statistical methods [7]. Iyer [6] discussed statistical procedures used in the calibration of measuring devices or measurement procedures. D'Errico [1] provided systematic treatment of statistical methods for measuring estimation. Fiset [5] analyzes operational methods for verifying the adequacy of mathematical models of multidimensional dynamical systems with biased measurements on the basis of statistical analysis of the sequences of residuals between outputs coordinates of systems and their models. The known measuring systems for contact forces measurement on the guide idlers are currently only based on theory [8].

3 Results and discussion

The test equipment of the pipe conveyor hexagonal idler housing consists of the 8 m long fixed transport belt. This equipment was described detailed in Molnár et al. [2]. So this is only a short description.

3.1 Test equipment

The measuring conception consists of two parts, tensioning part and idler housing part. The tensioning part is shown on Figure 1. The purposes of experiments realized at the equipment are based on change tension forces and analyzed contact forces changes with regard to the experiment goal. The tension forces are marked as $TF23$ and $TF24$ by the position number and abbreviated from TF, which means Tension Force, as shown on Figure 1. Total tension force, marked as $TFTotal$, is sum of $TF23$ and $TF24$.

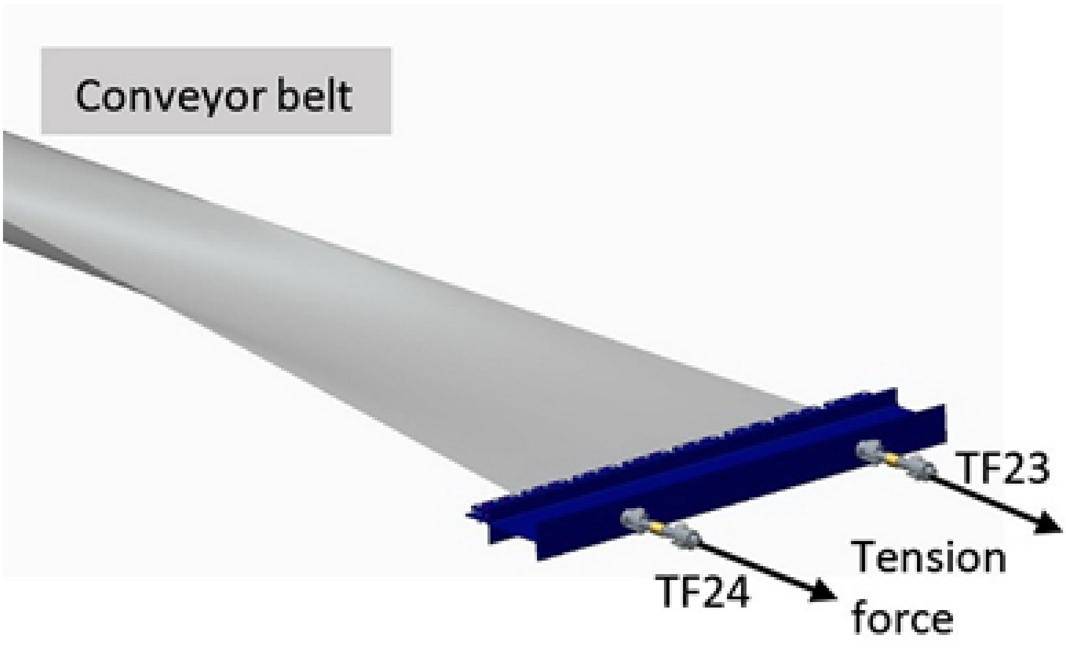

Figure 1: The tension force marking on the test equipment

Idler housing part of the equipment consists of the three hexagonal idler housing models. The contact force measurement is realized at each idler position. The first type of idler housing is that, which makes formation of the belt into the pipe shape. The second kind of the idler housing represents closed belt idler housings that is supporting and guiding the shaped piped belt. It represents most of the idler housings along the whole conveyor trajectory - see Figure 2. The third type of the idler housing is that kind, which is the last one before opening of the belt into the open, flat shape.

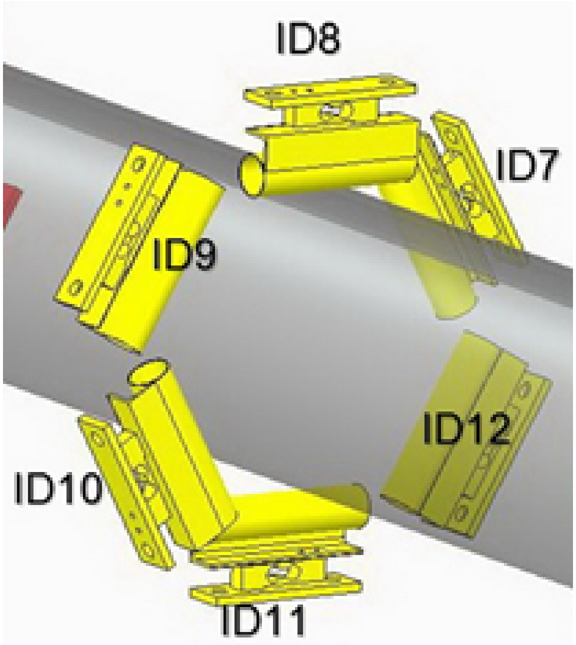

Figure 2: Marking positions of measured contact forces

The input variables of the experiment are the set amount of the tension force and the set parameters of the device. The output variable is the contact force marked by the position number and abbreviated from IDs that means idler. It is shown on Figure 2.

3.2 The principle and model of the experiment

The goal of the experiment is to identify whether there exists a significant effect of input variables changes in output variables that result in contact forces change.

Statistical model of the contact force formation is:

$$ID = f(TFTotal, Factor_j, Factor_{j+1}, \dots, Factor_k) + e.$$

Where:
ID is contact force value on the ID position, considered to be a functional relation.
$TFTotal$ is adjusted tension force.
$Factor_j, Factor_{j+1}, \dots, Factor_k$ can be interpreted as parameters of the device: kind of tension force setting, filling with material etc.
e is an error.

For purpose of analyses we named contact force changes as a variability. The variability of the measured values is caused either by the intentional changes or by the disruptive influences. The intentional input changes are presented and assigned to the variability source. The remaining variability in the contact force results is caused by disrupting inputs, at formula is named error. To obtain accurate results it is important to minimize variability from sources that are not part of experimental research. Eliminate of disrupting variability is a way to set tension force.

The experimental measurements were performed by two experimenters who both independently changed the tension force. Each experimenter set one tension force. The first experimenter set the wish value of $TF23$ and the second one set the wish value of $TF24$. The testing equipment was left without intervention for 60 s after each set-up process before the tension force was again changed.

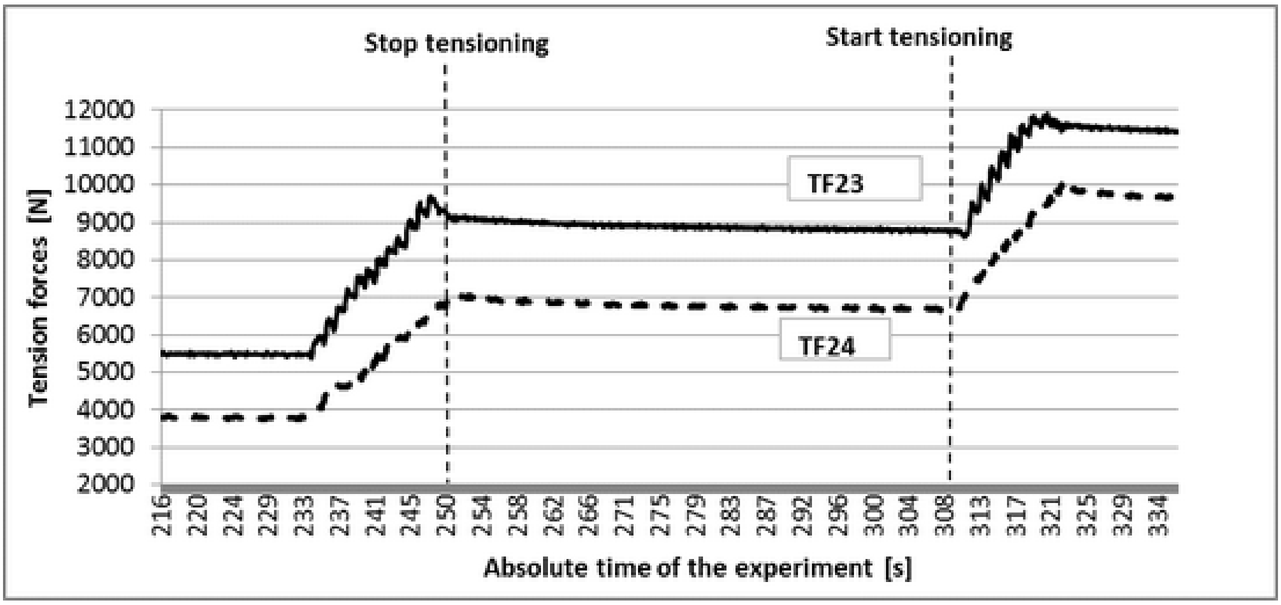

Figure 3: Tensioning and relaxating phase of the experiment

The facts known about the measuring equipment and the resulting consequences were taken into consideration for the determination of the methodology specified for the belt tensioning which was simultaneously performed manually by two service persons. For this reason it was not possible to keep a full synchronization of the time behavior during the process of the tension force increase. The model test equipment is static and after achieving the required tension force the system is stabilized. Jump movements of the transport belt on the idler rolls are typical for this stable state. The "relaxation creep" of the transport belt material is another Important attribute. Both aspects are influencing the contact force behavior. Figure 3 presents the behavior over the time of the tension forces for tension force settings of $TFTotal$ 12000 N, 18000 N and 24000 N. The horizontal axis is the absolute time of the experiment. The vertical axis depicts tension forces of $TF23$ and $TF24$. Figure 1 represents the end of the endurance with the tension force defined at the value $TFTotal = 12000$ N, it is $TF23 + TF24$. The next phase is tensioning. The vertical line "Stop tensioning" defines the time, which was necessary for achieving the tension force $TF23$, maximum value before the endurance phase. Tensioning at the $TF24$ was not finished at this time. The following time interval in the graph illustrates the endurance phase with the adjusted tension force value of 18000 N. It is possible to see a decrease of the $TF23$ and $TF24$ during this phase because of conveyor belt material properties. Some of the contact forces are increasing during the endurance phase while others are decreasing. The measuring process continues during the next tensioning phase. The beginning of this phase is marked with the vertical line marked "Start tensioning", which enables recording of a delay in the position $TF23$ in comparison to the position $TF24$.

The data used for our analysis are obtained from the relaxation phase compiled with the same time from beginning of the relaxation phase. This time is up to 30 s. The first vertical line in the graph "Stop tensioning", i.e. the end of the tensioning phase, is the time point 250 s in the absolute time of the experiment. The second vertical line "Start tensioing" - i.e. the end of the relaxation phase and beginning of a new tensioning phase, is the time point 308 s in the absolute time of the experiment. The difference between the lines is approx. 60 s. The data used for the analyses are from the middle part of the relaxation phase.

It is not possible to determine the exact starting point of the relaxation phase according to the records obtained from the measurements and determination of the starting point, using the graph is time demanding and subjective. It is clear that the measurement method causes variability that should be eliminated.

3.3 The propose a method of the phase analysis

A method of the phase analysis propose for our reason is based on the differences between the successive values. This method is founded on the reasonable assumption that the difference between the two successive values of the tension forces during the tensioning phase is positive.

The evaluated tension force value is the $TFTotal$. The calculation process, which used the given sampling frequency, was unable to determine the boundary of the tensioning phase uniquely. For this reason the difference between the successive values was calculated using the step 1 to 25 in the time interval from 233 s to 250 s in the absolute time of the experiment. This modification simulated a reduction of the sampling frequency. There was also a specified number of negative differences for each step during the given time interval - i.e. a number of wrong signals concerning finishing of the tensioning phase. The shortest possible step $D = 1.3$ was determined from the calculations and the wrong signals about finishing of the tensioning phase were eliminated in this way.

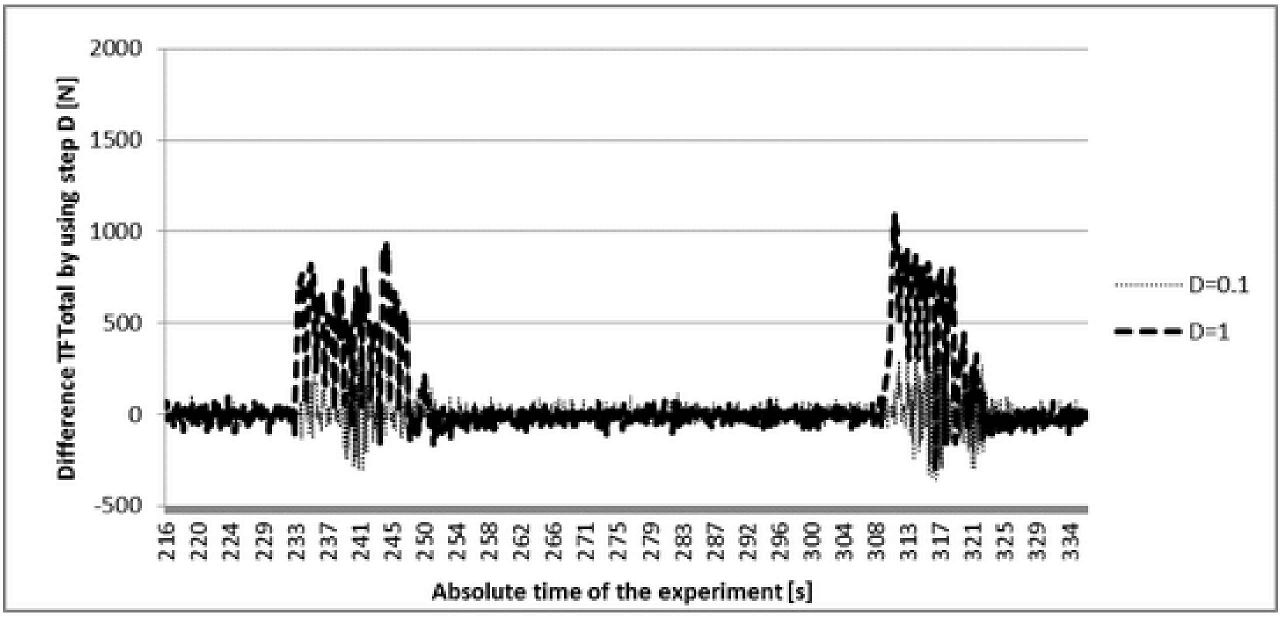

Figure 4: The difference $TFTotal$ with a step of difference $D = 0.1$ and $D = 1$

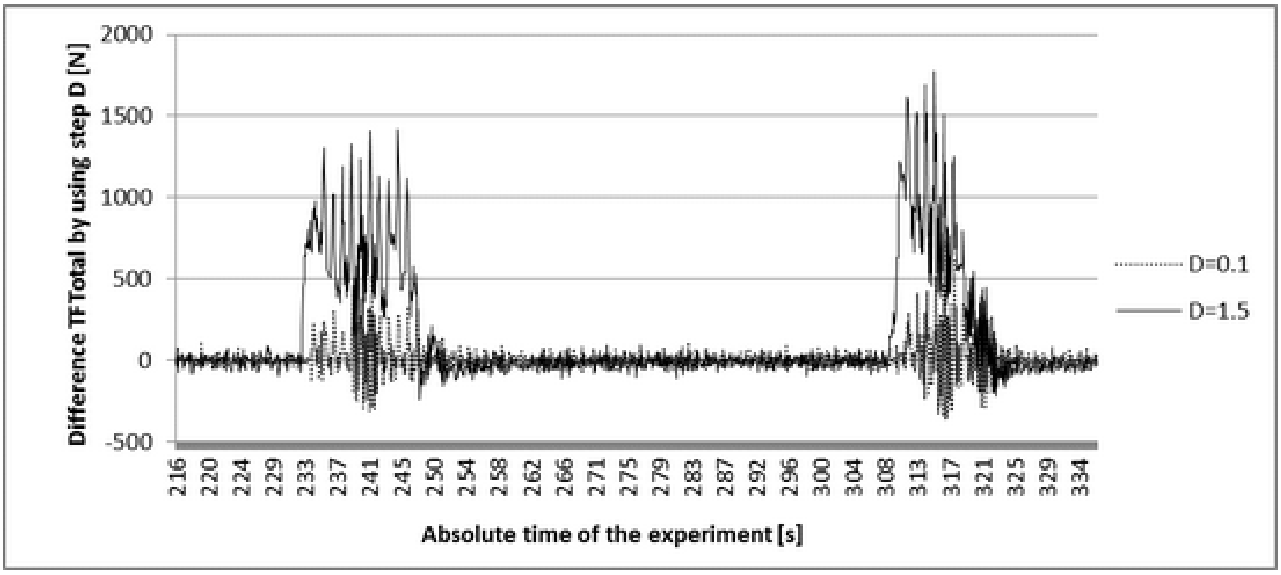

Figure 5: The difference $TFTotal$ with a step of difference $D = 0.1$ and $D = 1.5$

Figure 4 and Figure 5 illustrate the differences in the tensioning phase (transition of the curves below the x-axis) if the difference step is increasing. The step $D = 1.5$ (1.5 s) and the number 10 of successive negative signs of the signals were selected in order to determine the end of the tensioning phase during the time between the two tensioning phases - i.e. approx. in the middle of the interval.

The tension force differences are oscillating around zero for all steps and the system is sufficiently stable. This fact confirms a suitability of the adopted decision to use the data obtained in the time 30 s after the beginning of the relaxation phase.

4 Conclusion

In this paper we were interested in the method developed for the identification of the experiment phase (tensioning and relaxation), which is based on the differences of the contact force time behaviors. This method eliminates the variability of the measurement data.

Theoretical knowledge from research of the pipe conveyors are important in terms of the safety of their usage and economic efficiency. The complexity of the operating conditions, and the amount of uncontrolled inputs does not allow the realization of the operational measures which would bring possibilities for generalizations.

From the presented method, we expect the contingency of algorithms and the possibility of pre-processing of measured results in a simple manner without the need for manual calculations. It is uncertain whether the results of using the proposed method will be more useful for practice comparable to the old approach. Therefore, in the next step of the research, we will analyze the repeatability of the measurements. Within it, we aim to find out whether the method affects the accuracy and comparability of measurements.

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Bibliography

[1] D'Errico, G. E.: *Issues in Significance Testing*. Measurement. 42(10), (2009), 1478–1481.

[2] D'Errico, G. E.: *What Does Mean Mean? - A Systematic Treatment of Statistical Methods for Measurand Estimation*. Measurement. 43(4), (2010), 504–512.

[3] Dmitriev, V. G., Efimov, M. S.: *Estimation and Way of Decrease of Belt's Angular Deflection of a Tubular Conveyor at Rotary Movement*. Min. Inf. Anal. Bull. 16, (2009), 31–53.

[4] Drabiková, E., Fecková Škrabuľáková, E.: *Decision Trees - A Powerful Tool in Mathematical and Economic Modeling*. In: 18th International Carpathian Control Conference 2017 (ICCC 2017), Danvers, IEEE, 34–39.

[5] Fiset, M., Dussault, D.: *Laboratory Simulation of the Wear Process of Belt Conveyor Rollers*. In: 9th Internattional Conf. on Wear of Materials: San Francisco, CA: Apr. 13-16, 1993, WEAR, (162), 1012–1015.

[6] Iyer, H.: *Statistical Calibration and Measurements*. R. Khattree, C. R. Rao (Eds.), Handbook of Statistics, Statistics in Industry, vol. 22.

[7] Molnár, V., Fedorko, G., Stehlíková, B., Michalik, P., Kopas, M.: *Mathematical Models for Indirect Measurement of Contact Forces in Hexagonal Idler Housing of Pipe Conveyor*. Measurement. 47, (2014), 794–803.

[8] Molnár, V., Fedorko, G., Stehlíková, B., Tomašková, M., Hulínová, Z.: *Analysis of Asymmetrical Effect of Tension Forces in Conveyor Belt on the Idler Roll Contact Forces in the Idler Housing*. Measurement. 52, (2014), 22–32.

[9] Petrikova, I., Marvalova, B., Tuan, H. S., Bocko, P.: *Experimental Evaluation of Mechanical Properties of Belt Conveyor with Textile Reinforcement and Numerical Simulation of its Behaviour*. In: PConstitutive Models for Rubber VIII. Proceedings of the 8th European Conference on Constitutive Models for Rubbers, ECCMR 2013, (2013), 641–644.

[10] Sheng Chen, G., Lee, J. H., Narravula, V., Kitchin, T.: *Friction and Noise of Rubber Belt in Low Temperature Condition.* The influence of interfacial ice film, Cold Reg. Sci. Technol. 71, (2012), 95–101.

[11] Škrabuľáková, E., Marasová, D., Tomášková, M., Grendel, P.: *Identification of Factors Causing Damage to Conveyor Belts During the Transportation of Coal.* Applied Mechanics and Materials 683, (2014), Trans Tech Publications, Switzerland, 130–134.

[12] Woodcock, C. R., Mason, J. S.: *Belt Conveyors.* Bulk Solids Handling SE-7, (1988), 260–297.

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Cardinal invariant $\lambda(\mathcal{S},\mathcal{J})$

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Abstract: The cardinal invariant $\lambda(\mathcal{I},\mathcal{J})$ was introduced by J. Šupina in [7] as a combinatorial characteristics of $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\mathcal{J}\text{-}\Gamma)$-space. We analyze the cardinal invariant $\lambda(\mathcal{I},\mathcal{J})$ which we represent through slaloms, as was stated in [6]. We focus on three kinds of slaloms, namely h-slaloms, $\mathcal{A}$-slaloms and $\mathcal{I}$-slaloms, and we investigate the corresponding $\lambda(\mathcal{S},\mathcal{J})$ values. Finally, we summarize relations among mentioned cardinal invariants with respect to the well-known cardinal invariants of the continuum.

Keywords: Ideals, Slaloms, Cardinal invariant $\lambda(\mathcal{I},\mathcal{J})$.

Mathematics Subject Classification: 46A63, 37F20, 03E17.

1 Introduction

We study the cardinal invariant $\lambda(\mathcal{I},\mathcal{J})$ introduced by J. Šupina in [7] such that $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\mathcal{J}\text{-}\Gamma)) = \lambda(\mathcal{I},\mathcal{J})$, where $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\mathcal{J}\text{-}\Gamma)$ is a selection principle on a topological space and $\mathcal{I},\mathcal{J}$ are ideals on natural numbers ω, see [7] for further information. We deal also with its slight modification in the first parameter which we shall describe now.

Let $h \in {}^\omega\omega$ and $h(n) \geq 1$ for all but finitely many $n \in \omega$. The sequence s of finite subsets of ω is called an **h-slalom** if $|s(n)| \leq h(n)$ for each $n \in \omega$, see A. Blass [3]. We say that a function $\varphi \in {}^\omega\omega$

- **goes through** an h-slalom s if $\varphi(n) \in s(n)$ for all but finitely many $n \in \omega$,
- **evades** an h-slalom s if $\varphi(n) \in s(n)$ for finitely many $n \in \omega$.

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Let us recall a standard result by T. Bartoszynski [1] regarding slaloms as well. He has shown that

$$\mathrm{add}(\mathcal{N}) = \min\{|\mathcal{F}| :\ \mathcal{F} \subseteq {}^{\omega}\omega,\ (\forall\, \text{id-slalom } s)(\exists \varphi \in \mathcal{F})\ \neg(\varphi \text{ goes through } s)\},$$

where id obviously represents the identity. We shall study a cardinal invariant $\lambda(h, \mathrm{Fin})$ defined by

$$\lambda(h, \mathrm{Fin}) = \min\{|\mathcal{R}| :\ \mathcal{R} \text{ consists of } h\text{-slaloms},\ (\forall \varphi \in {}^{\omega}\omega)(\exists s \in \mathcal{R})\ \neg(\varphi \,\text{evades } s)\}.$$

The paper is divided into 6 sections. Main results are proved in the Section 2 which deals with the aforementioned cardinal invariant $\lambda(h, \mathrm{Fin})$ with respect to h-slaloms. In the third part, we discuss arbitrary families of slaloms and the corresponding invariant $\lambda(\mathcal{S}, \mathcal{J})$. The Section 4 discusses a cardinal invariant $\lambda(\mathcal{I}, \mathcal{J})$ having ideals in both parameters, which was originally introduced in [7]. We provide the combinatorial proof of $\lambda(\mathcal{I}, \mathcal{J})$ being monotone with respect to ordering of ideals here. Finally, we summarize obtained results to the diagram in the Conclusion. The last section called Appendix contains definitions and references to well-known cardinal invariants of notions appearing in the paper.

2 h-slaloms

In the present chapter, we focus on discussing the cardinal invariant $\lambda(h, \mathrm{Fin})$ for different functions h. Evidently, it is possible to compare cardinals $\lambda(h, \mathrm{Fin})$ with regards to the ordering of functions h. In particular, $\lambda(h, \mathrm{Fin})$ is decreasing with respect to the first parameter. Indeed, let $h_1, h_2 \in {}^{\omega}\omega$ be functions such that $h_1(n) \leq h_2(n)$ for each $n \in \omega$. Since each h_1-slalom is an h_2-slalom we obtain $\lambda(h_2, \mathrm{Fin}) \leq \lambda(h_1, \mathrm{Fin})$. Eventually, it is enough to suppose that $h_1(n) \leq h_2(n)$ for all but finitely many $n \in \omega$.

The following characterization showed by T. Bartoszyński [2] can be viewed as a degenerated case of the cardinal $\lambda(h, \mathrm{Fin})$, where h is a constant 1 function.[2]

$$\mathrm{non}(\mathcal{M}) = \min\{|\mathcal{F}| :\ \mathcal{F} \subseteq {}^{\omega}\omega,\ (\forall \varphi \in {}^{\omega}\omega)(\exists f \in \mathcal{F})\ |\{i :\ \varphi(i) = f(i)\}| = \aleph_0\}. \qquad (1)$$

Our goal is to prove that the cardinal invariant $\lambda(h, \mathrm{Fin})$ takes value $\mathrm{non}(\mathcal{M})$ for arbitrary function $h \in {}^{\omega}\omega$. Firstly, we begin with the bounded functions.

Lemma 1. *Let h be a bounded function such that $h(n) \neq 0$ for infinitely many $n \in \omega$. Then* $\mathrm{non}(\mathcal{M}) = \lambda(h, \mathrm{Fin})$.

Proof. The $\lambda(h, \mathrm{Fin}) \leq \mathrm{non}(\mathcal{M})$ is consequence of (1).

[2]We will denote such constant functions bolt writing in the chapter, for instance **1**.

Let $h \in \omega^\omega$ be a bounded functions and let $M = \max\{h(n) : \ n \in \omega\}$. Suppose that $\mathcal{R}$ is a family of h-slaloms witnessing the equality $|\mathcal{R}| = \lambda(h, \mathrm{Fin})$. Define functions f_i^s the following way: $f_0^s(n) = \max\{a : \ a \in s(n)\}$ for $i = 0$ and

$$f_i^s(n) = \begin{cases} f_{i-1}^s(n), & \text{if } s(n) \setminus \{f_0^s(n) \ldots f_{i-1}^s(n)\} = \emptyset, \\ \max\{a : \ a \in s(n) \setminus \{f_0^s(n), \ldots f_{i-1}^s(n)\}\}, & \text{otherwise,} \end{cases}$$

for each $0 < i < M$ and $s \in \mathcal{R}$. Let $\mathcal{F} = \{f_i^s : \ i < M \text{ and } s \in \mathcal{R}\}$. Thus $|\mathcal{F}| = |\mathcal{R}|$. Consider a function $\varphi \in {}^\omega\omega$. Then there is an h-slalom s from $\mathcal{R}$ such that $|\{n : \ \varphi(n) \in s(n)\}| = \aleph_0$. Finally, there is $i < M$ such that $|\{n : \ \varphi(n) = f_i^s(n)\}| = \aleph_0$ because M is finite. $\square$

It is clear that the inequality $\lambda(h, \mathrm{Fin}) \leq \mathrm{non}(\mathcal{M})$ also holds in a case of unbounded functions h. The reverse is not so obvious. The following statement presents $\lambda(h, \mathrm{Fin})$ takes the same value for any unbounded function h.

Lemma 2. *Let h_1, $h_2 \in {}^\omega\omega$ be unbounded functions. Then*

$$\lambda(h_1, \mathrm{Fin}) = \lambda(h_2, \mathrm{Fin}).$$

Proof. Let $h_1, h_2 \in {}^\omega\omega$ be unbounded functions such that $h_1(n) \leq h_2(n)$ for each $n \in \omega$. Thus $\lambda(h_2, \mathrm{Fin}) \leq \lambda(h_1, \mathrm{Fin})$. Since h_1 is the unbounded there exists an increasing function $\psi \in {}^\omega\omega$ such that $h_2(n) \leq h_1(\psi(n))$ for each $n \in \omega$ and then $\lambda((h_1 \circ \psi), \mathrm{Fin}) \leq \lambda(h_2, \mathrm{Fin})$. Now let $\mathcal{R}_2$ be a family of $h_1 \circ \psi$-slaloms such that $|\mathcal{R}_2| < \lambda(h_1, \mathrm{Fin})$. Define a family of h_1-slaloms

$$\mathcal{R}_1 = \left\{s^1 : \ s^1(n) = s^2(i), \text{ if } \exists i \ \psi(i) = n \text{ or } s^1(n) = \emptyset, \text{otherwise; for } s^2 \in \mathcal{R}_2\right\}.$$

Since $|\mathcal{R}_1| \leq |\mathcal{R}_2|$ there is φ_1 such that $\left\{n : \ \varphi_1(n) \in s^1(n)\right\}$ is finite for each $s^1 \in \mathcal{R}_1$. Consider a function $\varphi_2 = \varphi_1 \circ \psi$ and $s^2 \in \mathcal{R}_2$. Then

$$\left\{n : \ \varphi_2(n) \in s^2(n)\right\} \subseteq \left\{n : \ \varphi_1(n) \in s^1(n)\right\}$$

which is finite. Thus $\lambda(h_1, \mathrm{Fin}) \leq \lambda(h_2, \mathrm{Fin})$. $\square$

Finally, we prove the crucial lemma for the main result of this chapter. We focus on two specific functions, namely id and $\mathbf{1}$, which connect above facts.

Lemma 3. $\lambda(\mathbf{1}, \mathrm{Fin}) \leq \lambda(\mathrm{id}, \mathrm{Fin})$.[3]

Proof. Let $\langle I_n : \ n \in \omega\rangle$ be an interval partition of ω such that $|I_n| = n$ and $P_n = \left\{t : \ t \in {}^{I_n}\omega\right\}$ for any $n \in \omega$. Let ψ, $\varphi \in \prod_{n\in\omega} P_n$. We say that $\psi \preceq \varphi$ iff $\psi(n) \leq \varphi(n)$ as functions for each $n \in \omega$. Moreover, it is obvious that $({}^\omega\omega, \leq)$ and $(\prod_{n\in\omega} P_n, \preceq)$ are isomorphic. We will use them interchangeably without any other comments.

[3]After we sent the paper, D.A. Mejí advised an idea of the proof of the mentioned statement. Therefore, we decided with respect to our particular results to complete the proof and add it.

Let $\mathcal{S} \subseteq \left\{s \in \prod_{n\in\omega}[P_n]^{<\omega} :\ |s(n)| \leq n\right\}$ be a family of id-slaloms of the cardinality $\lambda(\mathrm{id}, \mathrm{Fin})$ which witnesses this property. We can enumerate members of $s(n)$ such that $s(n) = \{t_{n,k} \in P_n :\ k \in I_n\}$ for each $n \in \omega$ and $s \in \mathcal{S}$. Define a function y_s such that

$$\text{if } k \in I_n \text{ then } y_s(k) = t_{n,k}(k)$$

where $t_{n,k} \in s(n)$. Let $\mathcal{R} = \{y_s :\ s \in \mathcal{S}\}$. Thus $|\mathcal{R}| \leq |\mathcal{S}|$. Moreover, it is obvious that $\mathcal{R}$ is a family of $\mathbf{1}$-slaloms.

Let $f \in {}^{\omega}\omega$. Consider a corresponding function $\varphi \in \prod_{n\in\omega} P_n$ defined by $\varphi(n) = \{f(i) :\ i \in I_n\}$. By assumption there is an id-slalom $s' \in \mathcal{S}$ such that $|\{n \in \omega :\ \varphi(n) \in s'(n)\}| = \aleph_0$.

Finally for the corresponding $\mathbf{1}$-slalom $y'_s \in \mathcal{R}$, we have

$$\begin{aligned}\{n \in \omega :\ \varphi(n) \in s'(n)\} &= \{n \in \omega :\ \varphi(n) = t_{n,k} \text{ for some } k \in I_n\} \\ &\subseteq \{k \in \omega :\ f(k) = y_{s'}(k)\}.\end{aligned}$$

□

Theorem 4. $\lambda(h, \mathrm{Fin}) = \mathrm{non}(\mathcal{M})$ *for any* $h \in {}^{\omega}\omega$.

Proof. By the discussion before Lemma 1 and by (1) we have

$$\lambda(h, \mathrm{Fin}) \leq \lambda(\mathbf{1}, \mathrm{Fin}) = \mathrm{non}(\mathcal{M})$$

for any function h. By Lemma 2 $\lambda(\mathrm{id}, \mathrm{Fin}) = \lambda(h, \mathrm{Fin})$ for any unbounded function h. By Lemma 1 $\lambda(\mathbf{1}, \mathrm{Fin}) = \lambda(h, \mathrm{Fin})$ for any bounded function h. Hence, by Lemma 3 we can conclude $\lambda(h, \mathrm{Fin}) = \mathrm{non}(\mathcal{M})$ for any $h \in {}^{\omega}\omega$. □

3 $\mathcal{A}$-slaloms

The chapter discusses the value of $\lambda(\mathcal{A}, \mathcal{J})$, yet another modification of $\lambda(\mathcal{I}, \mathcal{J})$ for $\mathcal{A} \subseteq \mathcal{P}(\omega)$. We investigate the cardinal invariant with respect to different families in the first parameter. There is an easy observation mentioned in [6] which describes the taken values of $\lambda(\mathcal{A}, \mathrm{Fin})$ for very particular families $\mathcal{A}$. We offer the overview of that in more details.

Let $\mathcal{S} \subseteq {}^{\omega}\mathcal{P}(\omega)$. A sequence $s \in \mathcal{S}$ will be called a **slalom**. Let $\mathcal{J}$ be an ideal on ω. Recall a definition of $\mathcal{J}$-going of functions through slaloms which was used in [6]. A function $\varphi \in {}^{\omega}\omega$

- $\mathcal{J}$-**goes through** a slalom $s \in \mathcal{S}$ if $\{n :\ \varphi(n) \in s(n)\} \in \mathcal{J}^d$,
  i.e., $\{n :\ \varphi(n) \in \omega \setminus s(n)\} \in \mathcal{J}$,
- $\mathcal{J}$-**evades** a slalom $s \in \mathcal{S}$ if $\{n :\ \varphi(n) \in s(n)\} \in \mathcal{J}$.

In the same way as for h-slaloms, we say that φ goes through a slalom $s \in \mathcal{S}$ instead of φ Fin-goes through s and φ evades a slalom $s \in \mathcal{S}$, if Fin-evades s.

We shall define the cardinal invariant $\lambda(\mathcal{S}, \mathcal{J})$ as

$$\begin{aligned}\lambda(\mathcal{S}, \mathcal{J}) &= \min\{|\mathcal{R}| : \mathcal{R} \subseteq \mathcal{S}, (\forall \varphi \in {}^{\omega}\omega)(\exists s \in \mathcal{R})\ \neg(\varphi\, \mathcal{J}\text{-evades } s)\},\\ &= \min\{|\mathcal{R}| : \mathcal{R} \subseteq \mathcal{S}, (\forall \varphi \in {}^{\omega}\omega)(\exists s \in \mathcal{R})\ \{n : \varphi(n) \in s(n)\} \notin \mathcal{J}\}.\end{aligned}$$

It is obvious that aforementioned h-slaloms are particular case of slaloms with size additionally handled by function h. Therefore, there is no collision between notations $\lambda(h, \mathrm{Fin})$ and $\lambda(\mathcal{S}, \mathrm{Fin})$. In fact, $\lambda(h, \mathrm{Fin})$ is a particular case of $\lambda(\mathcal{S}, \mathrm{Fin})$ for family $\mathcal{S}$ of all finite slaloms bounded in size by the function h. Similarly as for h-slaloms, we can describe a monotonicity of $\lambda(\mathcal{S}, \mathcal{J})$ with respect to an inclusion. If $\mathcal{S}_1 \subseteq \mathcal{S}_2$ and $\mathcal{J}_1 \subseteq \mathcal{J}_2$ then $\lambda(\mathcal{S}_2, \mathcal{J}_1) \leq \lambda(\mathcal{S}_1, \mathcal{J}_2)$.

From now on, we will take care of different families $\mathcal{A}$ instead of $\mathcal{P}(\omega)$ in the definition of slaloms. Thus, let us stress that we use $\mathcal{A}$-slaloms to denote ${}^{\omega}\mathcal{A}$-slaloms and $\lambda(\mathcal{A}, \mathcal{J})$ instead of $\lambda({}^{\omega}\mathcal{A}, \mathcal{J})$.

We say that $\mathcal{A} \subseteq \mathcal{P}(X)$ has the **finite union property** if a complement of a union of any finite family $\mathcal{P} \subseteq \mathcal{A}$ is infinite, i.e., $|X \setminus \bigcup \mathcal{P}| = \aleph_0$. Note that there is an ideal which contains $\mathcal{A}$.

Proposition 5. *Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$ be such family that $\bigcup \mathcal{A} = \omega$.*

1. *If $\mathcal{A}$ has the finite union property and* $\mathrm{Fin} \subseteq \mathcal{A}$ *then* $\mathfrak{p} \leq \lambda(\mathcal{A}, \mathrm{Fin}) \leq \mathfrak{b}$.
2. *If $\mathcal{A}$ does not have the finite union property then*

$$\lambda(\mathcal{A}, \mathrm{Fin}) = \min\left\{k : \ \{A_0, A_1, \ldots, A_{k-1}\} \subseteq \mathcal{A} \text{ and } \bigcup_{i<k} A_i = \omega\right\}.$$

Proof. Since $\mathcal{A}$ has the finite union property there is an ideal $\mathcal{I}$ on ω such that $\mathcal{A} \subseteq \mathcal{I}$. Thus it holds $\mathfrak{p} \leq \lambda(\mathcal{I}, \mathrm{Fin}) \leq \lambda(\mathcal{A}, \mathrm{Fin})$. The first part of the inequality comes from $\lambda(\mathcal{I}, \mathrm{Fin}) = \min\{\mathtt{cov}^*(\mathcal{I}), \mathfrak{b}\}$ because both considered cardinals are at least $\mathfrak{p}$. On the other hand, since $\mathrm{Fin} \subseteq \mathcal{A}$ we have $\lambda(\mathcal{A}, \mathrm{Fin}) \leq \lambda(\mathrm{Fin}, \mathrm{Fin})$ which equals $\mathfrak{b}$. For both non-trivial relations see [6].

The second part is obvious. $\square$

For instance, let $\mathcal{A} = \{\{n\} : \ n \in \omega\}$. Then clearly, $\lambda(\mathcal{A}, \mathrm{Fin}) = \omega$. Moreover, one can easily observe that in the Proposition 5 the assumption $\bigcup \mathcal{A} = \omega$ is necessary.

4 $\mathcal{I}$-slaloms

Another specific families are ideals on ω. In fact, the $\lambda(\mathcal{I}, \mathcal{J})$ is the original notion introduced in [7] for ideals $\mathcal{I}, \mathcal{J}$. It was further investigated in [6].

In [7] there was mentioned the following result. It is a consequence of relations between $S_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-spaces and $\lambda(\mathcal{I}, \mathcal{J})$. We provide the direct combinatorial proof.

Proposition 6 (J. Šupina [7])**.** *Let* $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{J}_1$, $\mathcal{J}_2$ *be ideals on* ω *such that* $\mathcal{I}_1 \leq_{\mathrm{K}} \mathcal{I}_2$ *and* $\mathcal{J}_1 \leq_{\mathrm{KB}} \mathcal{J}_2$. *Then* $\lambda(\mathcal{I}_2, \mathcal{J}_1) \leq \lambda(\mathcal{I}_1, \mathcal{J}_2)$.[4]

Proof. We prove the proposition in two steps. Firstly, we show the inequality $\lambda(\mathcal{I}, \mathcal{J}_1) \leq \lambda(\mathcal{I}, \mathcal{J}_2)$ for any ideal $\mathcal{I}$.

Consider $\mathcal{R}_2 \subseteq {}^{\omega}\mathcal{I}$ such that $|\mathcal{R}_2| < \lambda(\mathcal{I}, \mathcal{J}_1)$. Since $\mathcal{J}_1 \leq_{KB} \mathcal{J}_2$ there is a finite-to-one function $\psi \in {}^{\omega}\omega$ such that $\mathcal{J}_1 \leq_{\psi} \mathcal{J}_2$. We define a family $\mathcal{R}_1$ which consisting of slaloms s^1 determined by $s^2 \in \mathcal{R}_2$ such that

$$s^1(i) = \begin{cases} \bigcup_{\psi(n)=i} s^2(n) & \text{if } \exists\, n\ \psi(n) = i, \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus $|\mathcal{R}_1| \leq |\mathcal{R}_2|$. By the assumption, there exists a function $\varphi_1 \in {}^{\omega}\omega$ such that $\{n :\ \varphi_1(n) \in s^1(n)\} \in \mathcal{J}_1$ for all $s \in \mathcal{R}_1$. Consider φ such that $\varphi(n) = \varphi_1(\psi(n))$ for $n \in \omega$. Then for an arbitrary slalom $s^2 \in \mathcal{R}_2$ we have

$$\begin{aligned} \{n :\ \varphi(n) \in s^2(n)\} &\subseteq \{n :\ \varphi_1(\psi(n)) \in s^1(\psi(n))\} \\ &= \psi^{-1}(\{n :\ \varphi_1(n) \in s^1(n)\}) \in \mathcal{J}_2. \end{aligned}$$

i.e., the function φ $\mathcal{J}_2$-evades each slalom $s^2 \in \mathcal{R}_2$.

In the second step we show that if $\mathcal{I}_1 \leq_K \mathcal{I}_2$ then $\lambda(\mathcal{I}_2, \mathcal{J}) \leq \lambda(\mathcal{I}_1, \mathcal{J})$ for an arbitrary ideal $\mathcal{J}$. Similarly as in the previous case, we consider a family $\mathcal{R}_1 \subseteq {}^{\omega}\mathcal{I}_1$, $|\mathcal{R}_1| < \lambda(\mathcal{I}_2, \mathcal{J})$. By the assumption, there is a function $\psi \in {}^{\omega}\omega$ such that $\mathcal{I}_1 \leq_{\psi} \mathcal{I}_2$. Define a family $\mathcal{R}_2$ by

$$\mathcal{R}_2 = \{s^2 :\ s^2(n) = \psi^{-1}(s^1(n)) \text{ for } s^1 \in \mathcal{R}_1\}.$$

Since $|\mathcal{R}_2| \leq |\mathcal{R}_1|$ there is a function $\varphi_2 \in {}^{\omega}\omega$ such that $\mathcal{J}$-evades each slalom $s^2 \in \mathcal{R}_2$. Let $\varphi \in {}^{\omega}\omega$ be a function such that $\varphi(n) = \psi(\varphi_2(n))$ for each $n \in \omega$. Hence for an arbitrary slalom $s^1 \in \mathcal{R}_1$ we have

$$\begin{aligned} \{n :\ \varphi(n) \in s^1(n)\} = \{n :\ \psi(\varphi_2(n)) \in s^1(n)\} &\subseteq \{n :\ \varphi_2(n) \in \psi^{-1}(s^1(n))\} \\ &= \{n :\ \varphi_2(n) \in s^2(n)\} \in \mathcal{J}. \end{aligned}$$

i.e., the function φ $\mathcal{J}$-evades each slalom $s^1 \in \mathcal{R}_1$. $\square$

5 Conclusion

In general, $\lambda(\mathcal{S}, \mathcal{J})$ can take several different values. Hence, there are at least two interesting questions coming: which values of known cardinal invariants could be taken and for which kinds of slaloms it could be taken.

[4] See Appendix for the definition of used orders of ideals.

In [6] it was shown that $\lambda(\mathcal{I}, \mathrm{Fin}) = \min\{\mathrm{cov}^*(\mathcal{I}), \mathfrak{b}\}$ for tall ideal $\mathcal{I}$. As [5] lists $\mathrm{cov}^*(\mathcal{I})$[5] values for known ideals we can easily determine their $\lambda(\mathcal{I}, \mathrm{Fin})$ value for some of them, see [6]. Particularly, the nowhere dense ideal[6] (denoted nwd) takes the value $\mathrm{cov}^*(\mathrm{nwd}) = \mathrm{cov}(\mathcal{M})$. Therefore, $\lambda(\mathrm{nwd}, \mathrm{Fin}) = \mathrm{add}(\mathcal{M})$, see [4, 5].

There was also shown that $\lambda(\mathrm{Fin}, \mathcal{J}) = \mathfrak{b}_{\mathcal{J}}$ in [7]. In particular, we obtain $\lambda(\mathrm{Fin}, \mathrm{Fin}) = \mathfrak{b}$. Moreover, we proved $\lambda(h, \mathrm{Fin}) = \mathrm{non}(\mathcal{M})$ for arbitrary function h. Finally, we summarize all obtained relations to the diagram.

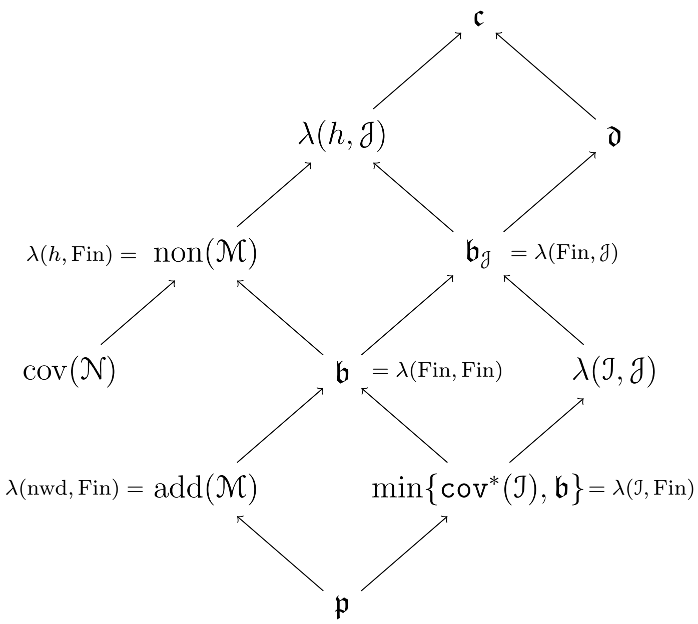

Figure 1: Cardinal invariants of the continuum and the $\lambda(\mathcal{S}, \mathcal{J})$.

Appendix

By an **ideal**[7] on ω we understand a family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ that is hereditary, i.e., $B \in \mathcal{I}$ for any $B \subseteq A \in \mathcal{I}$, closed under finite unions, contains all finite subsets of ω and $\omega \notin \mathcal{I}$. If not stated explicitly, ideal is an ideal on ω. Calligraphic $\mathcal{I}$, $\mathcal{J}$ are used to denote ideals. For $\mathcal{A} \subseteq \mathcal{P}(\omega)$ we denote

$$\mathcal{A}^d = \{A \subseteq \omega :\ \omega \setminus A \in \mathcal{A}\}.$$

Note that Fin is the ideal on ω consisting of all finite subsets of ω.

Apart from natural ordering of the pairs of all ideals by inclusion we can consider the following way. Let M_1, M_2 be infinite sets and $\mathcal{K}_1 \subseteq \mathcal{P}(M_1)$, $\mathcal{K}_2 \subseteq \mathcal{P}(M_2)$. If $\varphi :\ M_2 \to M_1$, the image of $\mathcal{K}_2$ is the family

$$\varphi^{\to}(\mathcal{K}_2) = \left\{A \subseteq M_1 :\ \varphi^{-1}(A) \in \mathcal{K}_2\right\}.$$

[5] $\mathrm{cov}^*(\mathcal{I}) = \min\{|\mathcal{A}| :\ \mathcal{A} \subseteq \mathcal{I} \wedge (\forall S \in [\omega]^\omega)(\exists A \in \mathcal{A})\ |S \cap A| = \omega\}$, i.e., the smallest cardinality of a subset of $\mathcal{I}$ that does not have a pseudounion, see [5].

[6] The ideal on the set of rational numbers $\mathbb{Q}$ whose elements are the nowhere dense subsets of $\mathbb{Q}$, see [5].

[7] Note that by ideal on ω we understand ideals defined on set of natural numbers as well as ideals on rational numbers and so on.

If $\mathcal{K}_2$ is an ideal on M_2 then $\varphi^{\rightarrow}(\mathcal{K}_2)$ is closed under subsets and finite unions and $M_1 \notin \varphi^{\rightarrow}(\mathcal{K}_2)$. If φ is in addition finite-to-one then $\varphi^{\rightarrow}(\mathcal{K}_2)$ is the ideal. For $\varphi : M_2 \to M_1$ we write

- $\mathcal{K}_1 \leq_\varphi \mathcal{K}_2$ if $\mathcal{K}_1 \subseteq \varphi^{\rightarrow}(\mathcal{K}_2)$, i.e., $\varphi^{-1}(I) \in \mathcal{K}_2$ for any $I \in \mathcal{K}_1$.

Then

- $\mathcal{K}_1 \leq_{\mathrm{K}} \mathcal{K}_2$ if there is a function $\varphi : M_2 \to M_1$ s.t. $\mathcal{K}_1 \leq_\varphi \mathcal{K}_2$,
- $\mathcal{K}_1 \leq_{\mathrm{KB}} \mathcal{K}_2$ if there is a finite-to-one function $\varphi : M_2 \to M_1$ s.t. $\mathcal{K}_1 \leq_\varphi \mathcal{K}_2$.

In the paper, we mentioned cardinal invariants of the continuum, like $\mathfrak{b}$, $\mathfrak{d}$, $\mathrm{add}(\mathcal{N})$, $\mathrm{add}(\mathcal{M})$, $\mathrm{cov}(\mathcal{N})$ or $\mathrm{non}(\mathcal{M})$. All of them are uncountable and at most equal to $\mathfrak{c}$. For more see any standard textbook, e.g. [3, 4]. Moreover, an ideal version of $\mathfrak{b}$, the invariant $\mathfrak{b}_\mathcal{J}$ can be found in [5].

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Bibliography

[1] Bartoszyński T.: *Additivity of Measure Implies Additivity of Category.* Trans. Amer. Math. Soc. 281, (1984), 209–213.

[2] Bartoszyński T.: *Invariants of Measure and Category.* Handbook of Set Theory, 281, (2010), 209–213.

[3] Blass A.: *Combinatorial Cardinal Characteristics of the Continuum.* Foreman M., Kanamori A. (eds.) Handbook of Set Theory. Springer, Dordrecht, (2010), 24–27.

[4] Bukovský L.: *The Structure of the Real Line.* Monogr. Mat., Springer-Birkhauser, Basel, 2011.

[5] Hrušák M., *Combinatorics of Filters and Ideals.* Contemp. Math. 533, (2011), 29–69.

[6] Šottová V., Šupina J.: *Principle* $\mathrm{S}_1(\mathcal{P}, \mathcal{R})$*: Ideals and Functions.* In: Topology Appl. 258, (2019), 282–304.

[7] Šupina J.: *Ideal* QN*-spaces.* J. Math. Anal. Appl. 434, (2016), 477–491.

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