Engineering Education in the 21st Century

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Editorial

Dear reader, you are opening a journal dedicated to new trends in engineering education - Engineering Education in the 21^{st} Century. The education reveals abilities but do not create them. Universities educate new generations that form the national elite. Well educated engineers became good workers and open-minded researchers whose creative ideas can reduce distances between continents. Therefore, scientific and pedagogical workers have to be devoted to their professions. It points out to the fact that educational activities and problems have their important place in nowadays scientific discussions.

Via this peer review proceedings we want to faster the research in fields of engineering education. The main mission of the eleven papers published in this journal is to help our readers in their hard and responsible work and give them the possibility to draw from them inspiration and encouragement. Their authors provide interesting and initiative ideas usable in the educational process as well as activities out of school.

We hope, that the obtained information encourage you in your study, education, teaching or research activity.

Editors

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Some didactic remarks connected with the concept of “proof” in mathematics

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***Abstract*—The teacher’s role in shaping students’ correct understanding of the elements of the methodology of mathematics is crucial and decisive. Many scientific works in the field of didactics of mathematics show that, unfortunately, the teaching of mathematics relating to a mathematical proof seems to be ineffective in many countries, regardless of how it is described in their educational documents and how this teaching is organized. In this paper one can find some didactic remarks connecting with the concept of “proof” in mathematics.**

***Index Terms*—Proof in Mathematics, Anthropomathematics, Teaching Mathematics**

I. Introduction

Reading and writing mathematical proofs are tasks that play a significant role in many upper division undergraduate mathematics courses. Unfortunately, many students encounter significant difficulties with the notion of rigorous proof (see, for example, [2]). The main objective of the work which has been presented here is to identify some problems of the teaching-learning process of mathematics, referring to one of the most important components of mathematical knowledge and skills of university students, such as the proving, explaining and justification, as necessary elements included in the mathematical methodology.

Review of the literature in the field of didactics of mathematics shows a continuous interest of the researchers in the subject oriented to the reasoning of the type of a proof in school mathematics. We are observing a steady increase in the number of research works aimed at the teaching and learning of a proof, regarding both theory and its practical applications in school. This subject is still up-to-date, which is expressed in still ongoing shifts of the topics related to this area in the plans and programs of teaching mathematics. As it can be guessed, these changes have two divergent directions. One trend is emphasis on intuitive reasoning without proofs in the teaching of mathematics. The educators and teachers proclaiming this view argue that due to the difficult conditions of teaching and learning of mathematics in the class-lesson system, a choice must be made between shaping students’ skills of proving and justifying, and developing their competence in solving mathematical problems, which policymakers consider to be more useful in human life. Such a view, if it is radical, treats proofs and proving as a hindrance to mathematical understanding, and not the way to this understanding [1].

Many scientific works in the field of didactics of mathematics show that, unfortunately, the teaching of mathematics relating to a mathematical proof seems to be ineffective in many countries, regardless of how it is described in their educational documents and how this teaching is organized. The difficulties that learners of mathematics have in that matter are of a diverse nature. It is stated, for example, that pupils and students:

- do not know the definitions required for proving of the concepts, they are not able to use them;
- they have ideas (also intuitive ones) of mathematical objects which are insufficient to perform proofs with their use;
- they are not able (or not willing) to generate and use their own examples;
- they are not able to understand the overall structure of a proof;
- they do not understand and are not able to use mathematical language and symbols;
- they do not know how to start proofs.

We can formulated some reasons for mentioned difficulties. One of them, which is very specific, is lack in students’ thinking the need to prove general statements. Such a deficit write some researchers in the field of mathematical education. For many teachers, as well as for many school and university students, a proof is just a “ritual” that occurs within mathematics. It is quite unnecessary in situations outside mathematics (see [5], [6]). In such situations, think students, a way to establish or verify a generalization could be: it is enough just to consider a few special cases. If these examples lead us to some generalization, then this generalization is certainly true. S. Vinner further states (see [6]) that he had an opportunity to observe such a way of thinking during classes in mathematics for graduate students preparing to become teachers in elementary schools. Namely, he said: we shall check how many subsets has a set consisting of n elements. Students worked together with the teacher, considering the cases $n = 1, 2, 3, 4, 5$. The group concluded that the sought number of subsets is 2^n. Then the teacher (S. Vinner) asked the students if they had any idea of how to prove it. I noticed - he writes - a look of surprise on their faces. One student said: *Aren’t the examples that we have considered sufficient to verify the truth of this generalization? Is it possible that* *this generalization was not true? And, to be quite honest, isn't a proof a superfluous formality?* I also noticed, writes S. Vinner (p. 29, [6]) that the other students nodded their heads in agreement as a sign that they fully agree with their colleague.

We must conclude that the teacher's role in shaping students' correct understanding of the elements of the methodology of mathematics is crucial and decisive. We should therefore pay particular attention to the education of students - future teachers, in this regard. These reflections and research observations (see [7], [8]) suggest the following conclusion: in the course of mathematical education, a future teacher develops his or her emotional attitude to the role of a proof in mathematics, and to the proving skills. This relation can be characterized by fear or reluctance caused by having a bad experience. It is possible to unconsciously transfer this negative attitude during the learning process (the phenomenon of "inheritance").

II. Research studies on students at university

The fact that it is necessary to improve the concept of teaching mathematics when it comes to developing proving skills, is confirmed by the example of a didactic research study, conducted on a group of first year students of didactic mathematics studies.

Two groups of first-year students were given some math problem in two versions to be solved:

The task "number of squares" - version A

Can the number of noticeable different squares in the following picture of a square with dimensions of n *by* n *be calculated using the following formula* $1^2 + 2^2 + ... + (n-1)^2 + n^2$*?*

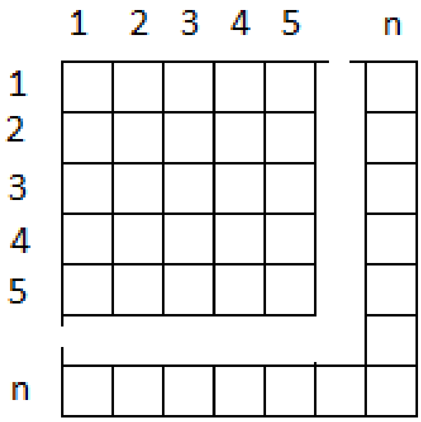

Fig. 1.

The task "number of squares" - version B

How many different squares can be seen in the following picture of a square with dimensions of n *by* n*, if* $n = 60$*?*

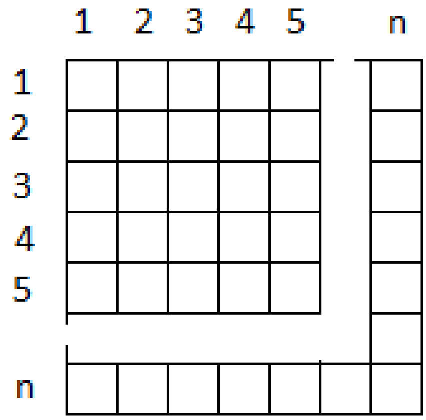

Fig. 2.

Versions A and B of this task differ in their instructions, and thus require a different kind of mathematical activity, however, both versions at some point require a verification of the hypothesis and its justification.

The task in version A was solved by 78 students (let us call this group SA), and in version B - 52 students (group SB).

The respondents from the group SB asked additional questions about the meaning of "different squares", the participants of the study from the group SA had no such doubts, perhaps even a preliminary analysis of the formula which they were given, clarified the issue.

Here is how students solved the task given to them.

Group SA

None of the students in the group SA questioned the truth of the verified hypothesis, and their typical procedure was as follows:

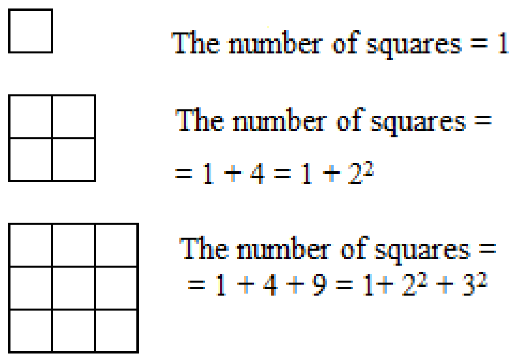

Fig. 3.

Some respondents, of course, skipped the check for $n = 1$. As many as 43 persons of the respondents after such verification finished solving the problem, sometimes adding a comment like: *it is obvious that it will always be like that*. Other students (a total of 23) carried out yet another test, for example, for $n = 4$ (Fig. 4):

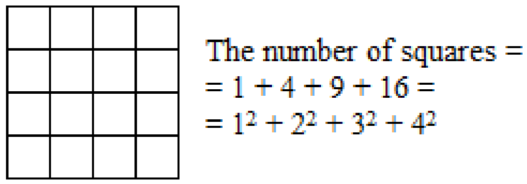

Fig. 4.

Few respondents (7 persons) verified this hypothesis for squares with a side length of 5,7, or even 10. It can be concluded that any subsequent, positively completed, attempt to verify the test hypothesis confirmed its reliability. Not many, because only 12 persons out of 78, attempted to carry out a mathematical proof of the hypothesis being verified. Perhaps, if the task instruction included the word “prove” - every student would take such an attempt, but the task description did not suggest it.

Group SB

The task in version B turned out for the surveyed students to be much more difficult. Its wording required primarily to put forward a hypothesis, to discover a method for counting squares and to apply it for $n = 60$. The use of a relatively large number n was, in my assumption, to inspire to put forward a hypothesis of a general nature which, with a small modification, would hold for any number $n \in N$. Perhaps it was the reason why as many as 14 people had not taken any attempts to solve the problem. Other persons were preoccupied with studying a specific case, usually a square with a side length of 4 (17 students) or 5 (21 students). Then, the persons solving the task applied different methods for counting squares, but only few of these methods led to a correct result. Only 6 surveyed students provided a general principle for counting the squares for $n = 60$, but they did not specify a general form of the hypothesis using the letter symbol (despite the presence of the symbol n), nor did they make explicit attempts to prove the correctness of the method of calculation which they made up. Some notes took a form suggesting that even though a student does not formulate a general hypothesis, nor justify it, his or her reasoning, however, can be regarded as an analysis of a concrete example based on the inference of a general nature, going beyond the concrete.

A similar study (using the same research tool) was conducted independently of my study, by G. Stylianides and A. Stylianides [3]. The data obtained in the study (of 39 students) also show that the only way of inference of students in the final year of the studies in didactic mathematics was the reasoning based on specific examples of the squares. Students formulated a hypothesis how to count the squares with the initial square side length 60 units, and they wrote:

$$1^2 + 2^2 + 3^2 + ... + 59^2 + 60^2.$$

When asked about the reasons for this method of calculation, they answered: *it can not be calculated, it can be calculated for a square with a side length of 4, maybe 5, and it already shows how it should be.*

The authors of the research studies conclude that for the whole survey group of students, a method of verifying the hypothesis was naive empiricism.

III. Anthropomathematical point of view

The results of the study described above imply that the first year students do not undertake independently and spontaneously any attempts to justify mathematical statements when they verify a ready hypothesis, or when they formulate it themselves. Such a need to justify general statements in mathematics probably is not a natural human goal. Feeling a need to justify general nature must be properly inspired - the inspiration can be instruction such as “prove” or “justify” in the mathematical task. Apparently, it seems that such an imperfect image of the role and importance of mathematical proof in practicing mathematics changes during mathematics studies. We expect that at this stage of mathematical education, students become convinced that mathematical proof is the sole and indisputable criterion of the truth of mathematical statements.

This problem lies within the research interests called by me *anthropomathematics*. The theoretical-research trend focused on an attempt to systematically characterize the “mathematical thinking through the prism of anthropomathematics”. This trend is consistent with what W. Thurston calls *human understanding mathematics* (see [4]). An accurate description of anthropomathematics approach to theory and practice teaching and learning mathematics can be found in my other works (see [7], [8]).

IV. Conclusion

One of the broad research areas is the issue of identifying the causes of so many problems associated with students’ understanding of mathematical methods, in particular with justifying general statements. This issue has been of interest to theoreticians and practitioners of mathematical education for a long time. However, although new attempts to answer the questions about the cause of students’ excessive difficulties and failures are continuously being made, it is still impossible to get unambiguous and constructive answers. Frequently, the cause of the occurrence of students’ difficulties is believed to be the deficiency in their knowledge and skills. Sometimes, those causes are sought on the teachers’ side, criticizing their methods of teaching. Sometimes, the excessive overload of thematic content in the curricula of mathematics is blamed. Probably, the truth lies somewhere in the middle, and an attempt to make a comprehensive diagnosis of this status quo requires a very broad and in-depth analysis of all the factors that may have some influence on this status quo. The analyses must be carried out here from different points of view and with a focus on deep understanding of the complexity of the process which we are dealing with.

One of these points of view is the anthropomathematical approach outlined in this work. It suggests:

• A significant **expansion of the analysis area of the process of teaching-learning mathematics**. The point is that the results achieved by students in the subject of mathematics should be looked at not only from the point of view of mathematical correctness of students' activities. The cognitive process which we are dealing with here, runs in a particular social system, the influence of which we must take into account.

• Consideration of **the influence of non-mathematical ways of thinking of man** on their reasoning carried out within the mathematical activity.

• Abandoning the image of mathematics as presented in teaching, as a field of infallible, absolute truths, and **conferring on mathematics a character of humanistic activity of man**. At the same time, due to the specific nature of mathematics and its complex epistemology, it should be assumed in advance that:

  ◦ the course and results of this activity may be subject to imperfection and prone to failure;

  ◦ the final results of this activity carried out by different people (within the same subject) can be varied as to their mathematical advancement.

And above all, a lot of research and considerations are required by the development of anthropomathematics as a research trend, crystallizing its research fields and planes and embedding them into international achievements, according to the principle of complementarity of didactic research studies.

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REFERENCES

[1] G. Hanna, *Proof, explanation and exploration: an overview*, Educational Studies in Mathematics, 44, (2000), 5–25.

[2] L. Sowder, G. Harel, *Case studies of mathematics majors' proof understanding, production, and appreciation*, Canadian Journal of Science, Mathematics and Technology Education, 3(2), (2003), 251–267.

[3] A. J. Stylianides, G. J. Stylianides, *Facilitating the transmision from empirical arguments to proof*, Journal for Research in Mathematics Education, 40(3), (2009), 314–352.

[4] W. P. Thurston, *On proof and progress in mathematics*, Bulletin of the American Mathematical Society, 30(2), (1994), 161–177.

[5] P. Tsamir, D. Tirosh, *Defining and proving with teachers: from preschool to secondary school*, In: B. Maj, E. Swoboda, K. Tatsis, (Eds.), *Motivation via natural differentiation in mathematics*, Wydawnictwo Uniwersytetu Rzeszowskiego, Rzeszów, 2010.

[6] S. Vinner, *Generalizations in everyday thought processes and in mathematical context*, In: B. Maj-Tatsis, K. Tatsis, (Eds.), *Generalization in mathematics at all educational levels*, Wydawnictwo Uniwersytetu Rzeszowskiego, Rzeszów, 2012.

[7] A. K. Żeromska, *Aksjologiczne konteksty edukacji matematycznej*, In: M. Kamińska -Juckiewicz, L. Tomaszewska, (Eds.), *Demokratyczne ścieżki edukacji*, Wydawnictwo PWSZ, Płock, 2010.

[8] A. K. Żeromska, *Metodologia matematyki jako przedmiot badań antropomatematycznych*, Wydawnictwo Naukowe UP, Kraków, 2013.

Existence proofs in discrete mathematics: constructive and probabilistic approaches

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***Abstract*—There are many investigation techniques which students and researchers may use in mathematics. According to the local conditions some of the techniques are preffered over the others. But all of them have their advantages and disadvantages and the research results may depend on the used investigation method. Therefore, it is good to know more of them. Here we compare the nonconstructive (probabilistic approach based on Lovász Local Lemma) with the classical constructive proofs. We describe pros and cons of both techniques and compare them at graph colouring examples.**

***Index Terms*—Constructive Proof, Nonconstructive Proof, Lovász Local Lemma, Thue Sequence, Nonrepetitive, Thue Type Problem, Gummi-bear Graph**

I. Introduction

There are several ways how to prove some graph properties in discrete mathematics. Naturally, not everybody can be familiar with all of them. Knowing about different investigation techniques is the promise of perspective movement forward in the research area. Therefore, it is good to show the students different angles from which one can view and possibly solve the same problem already during their education process.

In general, one can distinguish between constructive and nonconstructive approach. In mathematics, a constructive proof is a method of the proof that demonstrates the existence of a mathematical object by creating or providing a method for creating the object. This is in contrast to a nonconstructive proof which proves the existence of a particular kind of object without providing an example.
We will demonstrate the main difference between the constructive and nonconstructive proof on a simple example from algebra. Consider the following theorem:

Theorem 1. *There exist irrational numbers a and b such that $q = a^b$ is rational.*

Nonconstructive proof:
Choose appropriate irrational numbers a and b.
Let $a = b = \sqrt{2} \in \mathbb{I}r$. If $a^b = \sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ then we are done; else $a' = a^b = \sqrt{2}^{\sqrt{2}} \in \mathbb{I}r$, $b' = \sqrt{2} \in \mathbb{I}r$ and $a'^{b'} = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2 \in \mathbb{Q}$.

q.e.d.

This proof is nonconstructive. It relies on the statement "Either $q = a^b$ is rational or not".
The nonconstructive proof does not construct an example a and b but it gives a number of possibilities. By case analysis it showes that one of them has to yeld to the desired example. But it does not show which one.

Constructive proof:
Again we start with choice of appropriate irrational numbers a and b.
Let $a = \sqrt{2} \in \mathbb{I}r$, $b = \log_2 9$. Then $a^b = 3 \in \mathbb{Q}$. While the fact that $\sqrt{2} \in \mathbb{I}r$ is well known, it is not that obvious that also $\log_2 9 \in \mathbb{I}r$. Therefore, we write a proof of it as a supplement of the proof of the main statement:
If $\log_2 9$ would be rational number equal to $\frac{m}{n}$, then $n \cdot \log_2 9 = m$ and $\log_2 9^n = m = \log_2 2^m$. Hence, we have that 9^n should be equal to 2^m. But it is impossible, as one of these numbers is even and the second one is odd. Hence, $\log_2 9 \in \mathbb{I}r$.

q.e.d.

Constructive and nonconstructive proofs are of use not only in algebra and number theory. In this paper we also give some examples to demonstrate how both kinds of proofs work in discrete mathematics. Graph theory is a part of discrete mathematics where we very often deal with determination of some graph parameter. Sometimes we are unable to determine the exact value of considered graph parameter. In such situations we try to get some upper and lower bounds for it. Naturally, we want them being as good as possible. Results in the form of upper and lower bounds may differ according to chosen investigation method. Therefore, it is good to know about the existence of more methods. Of course, each of these methods has both advantages and disadvantages. Here we compare the nonconstructive - probabilistic approach based on Lovász Local Lemma, with the classical constructive proofs. We describe pros and cons of both techniques and compare them at graph colouring examples, focussing on the graph colouring problems of Thue type.

II. Constructive proofs

Constructive proofs in graph theory are usually of use at those situations when we investigate some problem on finite family of graphs or we have some good information about the structure of graphs even from infinite family.

If the researched problem is of the type: *Find the minimum number of colours m such that using m colours to colour the vertices (edges) of the given graph G the graph colouring is of property A*, than finding a colouring φ of G with k colours that fulfills A means that we have found only the upper bound for the investigated graph parameter. To prove that the obtained bound for the concrete parameter is exact value of this parameter means that we need to prove that using fewer colours to colour the vertices (edges) of the graph G necessary effect that the colouring will not have the property A.

If the researched problem is of the type: *Find the maximum number of colours M such that using M colours to colour the vertices (edges) of the given graph G the graph colouring is of property B*, than finding a colouring χ of G with ℓ colours that fulfills B means that we have found only the lower bound for the investigated graph parameter. To prove that the obtained bound for the investigated parameter is exact value of it in this case means that we need to prove that using more colours to colour the vertices (edges) of the graph G necessary effect that the colouring will not have the property B.

In both of the cases it is often appropriate to use the indirect proof: We consider that we can use fewer (more) colours to colour the graph G such that it will have a property A (B respectively) as by given colouring φ (χ respectively) but we came to a contradiction.

A. Graph colouring problems of Thue type

Graph colouring problems are typical problems where both constructive and nonconstructive proofs are of use. Since the formulation of the "Four Colour Problem" many questions concerning colourings of graphs aroused. With a computer-science development also the research on string-type chains became more and more popular. This is connected with graph colouring problems of Thue type[1], that deal with nonrepetitive sequences. Here by a repetition we mean such a sequence, in which there exist two consecutive identical blocks of terms. On the other hand, nonrepetitive sequences are the sequences where no two consecutive blocks are the same. Mathematically, *repetitive* sequence or *repetition* is a sequence $r_1, r_2, \ldots, r_{2n}$ such that $r_i = r_{n+i}$ for all $1 \leq i \leq n$. A sequence S is called *nonrepetitive* if no subsequence of consecutive terms of S is a repetition. E. g. the words BARBAR or ILLNESS contain a repetition (are repetitive) while PALM or MINIMUM do not (these words are nonrepetitive).
We say that a sequence is *rainbow* if it consists of pairwise different terms.

[1]Axel Thue was Norwegian mathematician who started systematic study in structures of words and studied basic objects of theoretical computer science long before the invention of the computer or DNA. One of remarkable discoveries made by Thue is the fact, that consecutive repetitions of non-empty subwords can be avoided in infinite words over three letter alphabet [30].

Let $G = G(V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A sequence of consecutive vertices and edges $v_1 e_1 v_2 e_2 \ldots v_n$ of G, where for $i \neq j$: $v_i \neq v_j$, is called a path P_n in G. By the *length* of a path we mean the number of its edges, i.e., a path P_n on n vertices has length $n-1$.

An *vertex k-colouring* of G is a mapping $\phi : V(G) \to \{1, 2, \ldots, k\}$. A vertex-colouring ϕ of a graph G is *non-repetitive* if the sequence of colours on any path in G is nonrepetitive. The minimum number of colours $\pi(G)$ needed in a nonrepetitive vertex-colouring of G is nowdays[2] called the *Thue chromatic number* of G.
Sometimes it is possible to use only a colour from some list preassigned to each vertex of G, when colouring its vertices. In such a case we speak about a list colouring of G. For the case of list vertex-colouring the *Thue choice number* $\pi_\ell(G)$ of a graph G denotes the smallest number k such that for every list assignment $L : V(G) \to 2^{\mathbb{N}}$ with minimum list length at least k there is a colouring of the vertices from the assigned lists such that the sequence of vertex colours of no path in G forms a repetition.

We call two vertex k-colourings φ and ψ *equivalent* if there is a permutation σ of the colour set $\{1, ..., k\}$ and an automorphism δ of the graph G, such that $\psi = \sigma \circ \varphi \circ \delta$. Basically, this means that one colouring is derived from the other by relabeling of colours and vertices. Clearly, for two equivalent k-colourings φ, ψ of a graph G we have:

$$\varphi \text{ is a non-repetetive colouring of } G$$
$$\Updownarrow$$
$$\psi \text{ is a non-repetetive colouring of } G$$

Other notation and terminology used but not defined here can be found in [4].

B. The proofs

Sometimes an observation that some graph-structure has "somehow unique" colouring with the property P is the key observation on which is based the whole constructive proof. We will demonstrate it at one example - the proof that gummi-bear graphs are a family of graphs where the Thue chromatic number differs from the Thue choice number: $\pi(G) \neq \pi_\ell(G)$.

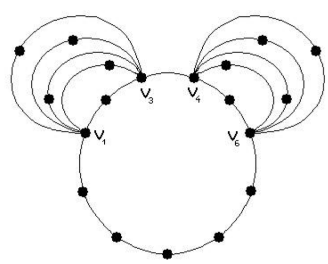

Fig. 1. The smallest example of a gummi-bear graph

[2]see the historical remark about notations in [28]

Definition 1. *A gummi-bear graph is a graph consisting of a cycle on* 11 *vertices* $v_1, v_2, \ldots, v_{11}$ *in this cyclic order and a set of "ears" - at least* 4 *additional paths of length* 2 *connecting the vertices* v_1, v_3 *and at least* 4 *additional paths of length* 2 *connecting the vertices* v_4, v_6.

The smallest example of a gummi-bear graph has 19 vertices and is depicted on Figure 1.

The vertices v_1, v_3, v_4, v_6 are called the *end-vertices of some ear*, the cycle on the vertices $v_1, v_2, \ldots, v_{11}$ is called the *main cycle* of the gummi-bear graph and all vertices not belonging to the main cycle are called *ear-vertices* or *vertices on ears*.

To show that the Thue chromatic number differs from the Thue choice number in the family of gummi-bear graphs, we will need to show that the nonrepetitive 3-colouring of vertices of the main cycle of a gummi-bear graph is "unique" up to the equivalence defined above. As the proof of the "uniquess" of colouring of some graph substructure needs some longer discussion, we will proof it rather separately as a lemma.

Lemma 1. *Up to equivalence there is exactly one nonrepetitive* 3*-colouring of the cycle* C_{11}.

Proof:
Let C_{11} be a the cycle on 11 vertices. We denote the vertex set by $V = \{v_1, ..., v_{11}\}$ and the edge set by $E = \{v_iv_j : |i-j| = 1\} \cup \{v_{11}v_1\}$. Let φ be a nonrepetitive 3-colouring of C_{11}. First assume, that every 3 consecutive vertices are pairwise distinctly coloured. Then up to equivalence there has to be a path P on 6 vertices in C_{11} with colour sequence 123123 which is a repetition - a contradiction.
Hence, there has to be a sequence of 3 consecutive vertices, such that two of them get the same colour by φ. Up to equivalence this means $\varphi(v_1) = \varphi(v_3) = 1$ and $\varphi(v_2) = 2$. To avoid a repetition it follows $\varphi(v_4) = 3$. Now we construct the sequence $\{\varphi(x_i)\}_{i=1}^{11}$ starting from the subsequence 1213. Figure 2 illustrates the case study. After each subsequence we have at most two possibilities how to proceed, because consecutive numbers have to be distinct. In the possibility tree in Figure 2 a number is stroked from the upper right to the lower right corner if a repetition is constructed by choosing this number as a colour for the corresponding vertex. In the last column the numbers 2 and 1 are stroked because when $\varphi(v_{11}) \neq 3$, we will came to a repetition on the main cycle C_{11}.

As a result we obtain that up to equivalence there exist at most four different nonrepetitive colour sequences using colours 1, 2 and 3 - written on the right part of Figure 2.
We claim, that all four sequences represent equivalent colourings. To show this we will take a colour permutation for each of the first three sequences, then cyclically shift the obtained colour sequences until they start with 1213 and in each case we obtain the fourth sequence.

- First sequence: exchange 1 and 3 to receive 32313213121 and shift the sequence by 8.

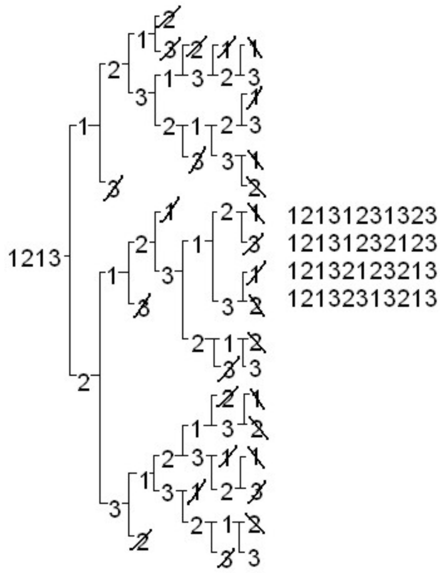

Fig. 2. Nonrepetitive sequences

- Second sequence: exchange 2 and 3 to receive 13121323132 and shift the sequence by 2.
- Third sequence: cyclically shift the colours $(1 \mapsto 3, 2 \mapsto 1, 3 \mapsto 2)$ to receive 31321312132 and shift the sequence by 6.

This proves the claim. Hence, up to equivalence there is at most one nonrepetitive 3-colouring of C_{11} represented by any of the sequences in Figure 2. It is easy to check, that the respective colouring is indeed a nonrepetitive 3-colouring of the C_{11}. Note that no matter in which orientation we consider the colouring of some the cycle, if the colouring is repetitive, the repetition will occur in both directions. Hence, to check whether some repetition occurs on the cycle it is not necessary to consider both possible orientations.

q.e.d.

An easy observation about nonrepetitive sequences is the following: If a nonrepetitive sequence is interrupted by nonrepetitive sequences using a distinct set of symbols, then the resulting new sequence remains nonrepetitive. This was formally proved in [14].

Now we finally prove the theorem:

Theorem 2. *There exists an infinite class of graphs where each graph from the class is nonrepetitively* 3*-colourable but not nonrepetitively* 3*-choosable.*

Proof:
Consider any gummi-bear graph G. At first we show that G is nonrepetitively 3-colourable. For that purpose consider the subgraph H of G, where one ear is added to the main cycle on each side. The main cycle itself has to be coloured nonrepetitively. According to Lemma 1 up to equivalence there is a unique such 3-colouring. This corresponds to eleven non-

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
i	**1**	2	**1**	**3**	2	**3**	1	3	2	1	3
ii	**3**	1	**2**	**1** $\downarrow$	3 $\leftrightarrows$	**2** $\rightarrow$	3 $\rightarrow$	1 $\rightarrow$	3 $\rightarrow$	2	1
iii	**1** $\leftarrow$	3 $\leftarrow$	**1** $\leftarrow$	**2** $\leftarrow$	1 $\rightleftarrows$	**3** $\downarrow$	2	3	1	3	2
iv	**2** $\downarrow$	1 $\leftrightarrows$	**3** $\rightarrow$	**1** $\rightarrow$	2 $\rightarrow$	**1** $\rightarrow$	3	2	3	1	3
v	**3**	2	**1**	**3**	1	**2**	1	3	2	3	1
vi	**1** $\leftarrow$	3 $\rightleftarrows$	**2** $\downarrow$	**1**	3	**1**	2	1	3 $\leftarrow$	2 $\leftarrow$	3 $\leftarrow$
vii	**3**	1	**3**	**2** $\downarrow$	1 $\leftrightarrows$	**3** $\rightarrow$	1 $\rightarrow$	2 $\rightarrow$	1 $\rightarrow$	3	2
viii	**2**	3	**1**	**3**	2	**1**	3	1	2	1	3
ix	**3** $\leftarrow$	2 $\leftarrow$	**3** $\leftarrow$	**1** $\leftarrow$	3 $\rightleftarrows$	**2** $\downarrow$	1	3	1	2	1
x	**1** $\downarrow$	3 $\leftrightarrows$	**2** $\rightarrow$	**3** $\rightarrow$	1 $\rightarrow$	**3** $\rightarrow$	2	1	3	1	2
xi	**2** $\leftarrow$	1 $\rightleftarrows$	**3** $\downarrow$	**2**	3	**1**	3	2	1 $\leftarrow$	3 $\leftarrow$	1 $\leftarrow$

TABLE I
NONREPETITIVE 3-COLOURINGS OF C_{11}

Case	**Repetition**
I	11, 2121 or 3131
II	31323132
III	13121312
IV	12131213
V	
VI	32313231
VII	12131213
VIII	
IX	32313231
X	31323132
XI	13121312

TABLE II
REPETITIONS THAT APPEAR IN SITUATIONS DESCRIBED IN TABLE I AFTER BOTH MAIN CYCLE AND EARS ARE COLOURED.

equivalent possibilities how to colour the vertices v_1, v_2, v_3, $v_4, \ldots, v_{11}$ of the main cycle C_{11} of the graph H - see Table I, where the colours of the endvertices of the ears are written bold.

Note that to obtain a nonrepetitive 3-colouring of H every time when two different colours are used on the end-vertices of some ear, the corresponding ear-vertex has to be coloured with the third colour. But then some repetition (see Table II) will occur in cases II–IV, VI–VII and IX–XI. To find the corresponding paths, start with the ear vertex which has the same neighbours as the vertex with the double arrow and follow the arrows in Table I.
Moreover, it is easy to find a repetition in the case I too: If we colour the ear-vertex of the ear with end-vertices v_1, v_3 with colour 1 or 2, the repetition of the form aa or $abab$ will occur. If we colour it 3, the repetition 3131 starting with the colour of this ear-vertex and continuing with the colours of the vertices v_1, v_{11}, v_{10} will occur.

In the cases V and VIII two different colours are used on the end-vertices of both ears. Hence, the ear-vertex on the ear with end-vertices v_i, v_{i+2}; $i \in \{1, 4\}$, has to obtain the same colour as the vertex v_{i+1}. Then a repetition will not occur on any cycle of length 4 and obviously not on the main cycle. Hence, if there is a repetition in the graph H, then it is a repetition on some path, which is not a subpath of the main cycle. In case V it must have been a subpath of one of the paths coloured 23213121, 232131213231, 2123132312131, 121312313231. But as all of them are nonrepetitive, we have a nonrepetitive 3-colouring of the graph H. Similarly, the repetitive path in case VIII can only be a subpath of one of the paths coloured 32313212, 323132131213, 3132312131232, 212313231213 all of which are nonrepetitive, hence, we have again the nonrepetitive 3-colouring of the graph H. Moreover, both colourings can be easily extended to a nonrepetitive 3-colourings of any gummi-bear graph by assigning the same colour to each additional ear, as was chosen for the first ear. Obviously, a repetitive path in the gummi-bear graph G can only exist if there was one in H.

Now we show that gummi-bear graphs are not nonrepetitively 3-choosable. We prove the statement by contradiction. If a gummi-bear graph G is nonrepetitively 3-choosable, then this also holds for the subgraph G' consisting of the main cycle and four ear vertices x_1, x_2, x_3, x_4 with end vertices v_1 and v_3 and four ear vertices y_1, y_2, y_3, y_4 with end vertices v_4 and v_6. Now consider the special list assignment L where all vertices on the main cycle have the list $\{1, 2, 3\}$ and $L(x_i) = L(y_i) = \{1, 2, 3, 4\} \setminus \{i\}$ for $i \in \{1, 2, 3, 4\}$. Assume that G' is nonrepetitively colourable with colours from the lists of L. Then up to renaming of colours, the subgraph H induced on vertices of the main cycle, x_4 and y_4 must be coloured according to the colouring from case V or VIII. Regardless how the colours 1, 2, 3 are permuted, all additional ear vertices between v_1 and v_3 must receive the same colour as v_2 or colour 4, otherwise there will be a repetition of the form aa. Similarly, the additional ear vertices between v_4 and v_6 must receive the same colour as v_5 or the colour 4. Because each colour i from $\{1, 2, 3\}$ is missing in at least one list of one of the ears of each side (in $L(x_i)$ and $L(y_i)$), it is clear that the colour 4 must appear at least once on an ear vertex x between v_1 and v_3 and at least once on an ear vertex y between v_4 and v_6. But then a repetition of the form $4ab4ab$ occurs in both cases ($P_V = (y, v_3, v_2, x, v_1, v_{11})$ and $P_{VIII} = (x, v_3, v_4, y, v_6, v_7)$, respectively), a contradiction.

q.e.d.

Here we showed that there exists an infinite class of graphs in which Thue chromatic number of each graph from the family is not equal to the Thue choice number of that graph, even if the difference is minimal. But it is even possible to prove that the difference between the Thue chromatic number and the Thue choice number in some other families of graphs

(e. g. trees) can be arbitrary large [8].

The proof of Theorem 2 shows that the constructive proof might work also in the case when we are dealing with list colourings. This is not a standart case, as it is not easy to operate on a random phenomena and prove something about it constructively.

Some more constructive proofs of theorems dealing with problems of Thue type can be find in [1], [3], [5], [9], [12], [13], [14], [16], [17], [19], [20], [24] and elsewhere.

Constructive proofs in graph theory are realy effective mostly in the cases when we know a lot of features of the graph or graph structure.
The big advantage of constructive proof is that it is easy readable and usually it requires not that much background from other mathematical disciplines. University students prefer such proofs, even if sometimes these proofs require some case-analysis, as we have provided in the proof of Theorem 2 - our example. The case analysis makes the proofs from time to time too long, what is disadvantage. One has then decide whether the length of the proof will not effect its clarity and whether then some other kind of proof would not be more appropriate.

III. Probabilistic proofs

Sometimes it is not known too much about the structure of a graph or family of graphs we are investigating, respectively, family of graphs we are focusing on is very rich from the structural point of view, or it is infinite. In such a case it is more suitable to use something else than constructive approach to prove that the graph, or family of graphs has the desired property. It makes no sense to take one graph after another and investigate it in order to prove some graph property. However surprising it sounds, in these cases often some knowledge from probability theory is of use.

Probability theory deals with the analysis of random phenomena. To say that two events are independent means, that the occurrence of one does not affect the probability of the other. Similarly, two random variables are independent if the observed value of one does not affect the probability distribution of the other.
Two events A and B are independent iff their joint probability $P(A \cap B)$ equals the product of their probabilities: $P(A \cap B) = P(A)P(B)$. Hence, the occurrence of B does not affect the probability of A, and vice versa. We can rewrite it via conditional probabilities:

$$P(A \cap B) = P(A)P(B)$$

$$\Updownarrow$$

$$P(A) = \frac{P(A \cap B)}{P(B)} = P(A|B)$$

We need not necessary to deal with two events. The concept of independence should be extended for a collection of events or random variables: A finite set of events $\{A_i\}$ is pairwise independent iff every pair of events is independent. Then

$$P(\cap_{k=1}^{n} A_k) = \prod_{k=1}^{n} P(A_k \mid \cap_{j=1}^{k-1} A_j).$$

A. Lovász Local Lemma

If a large number of events in probability theory are all independent of one another and each has probability less than 1, then there is a positive (possibly small) probability that none of the events will occur. The Lovász Local Lemma - a very powerfool tool based on which also a lot of proofs in graph theory are given, allows one to relax the independence condition as follows: As long as the events are "mostly" independent from one another and aren't individually too likely, then there will still be a positive probability that none of them occurs. It is most commonly used in the probabilistic method, in order to give existence proofs. There are several different versions of the lemma (see [2]). The simplest and most frequently used is the symmetric version given below:

Let A_1, A_2, $\ldots$, A_k be a series of events such that each event occurs with probability at most p and such that each event is independent of all the other events except for at most d of them. In 1975 Erdős and Lovász showed [6] that: *If*

$$4 \cdot p \cdot d \leq 1,$$

then there is a nonzero probability that none of the events occurs.
In 1977 Joel Spencer published [29] an improvement (for $d > 2$) to this result: *there is a nonzero probability that none of the events occurs already if*

$$e \cdot p \cdot (d + 1) \leq 1,$$

where e = 2.718... is the base of natural logarithms. And this is the version that is today reffered as *Lovász Local Lemma*:

Lemma 2. *Let $A_1, A_2, \ldots, A_n$ be a set of "bad" events with $P(A_i) \leq p < 1$ and each event A_i is mutually independent of all but at most d of the other A_j. If $e \cdot p \cdot (d + 1) \leq 1$, then $P(\cap_{i=1}^{n} \overline{A_i}) > 0$.*

The proof of this and other versions of the Lovász Local Lemma can be find in diverse scientific literature e.g. [2], [6], [7], [15], [18], [21] [22], [23] and elsewhere.

B. Proofs based on Lovász Local Lemma

A simple example of usuage of the symmetric Lovász Local Lemma is the proof of the following graph-colouring Theorem:

Theorem 3. *Let C_{11n} be a cycle on $11n$ vertices which are colored with n different colors in such a way, that each color is applied to exactly 11 vertices. In any such coloring, there is a set of n vertices containing one vertex of each color but not containing any pair of adjacent vertices.*

Proof:
The random choice of picking a vertex of each color has a

probability $\frac{1}{11}$ as all vertices are equally likely to be chosen. We have $11n$ different events that correspond to the $11n$ pairs of adjacent vertices on the circle. These events we want to avoid. For each pair of adjacent vertices on the cycle the probability of picking both vertices in that pair is at most $\frac{1}{121}$ (If the two vertices are of different colors, it is exactly $\frac{1}{121}$, otherwise it is 0), so we will take $p = \frac{1}{121}$. Whether a given pair (a, b) of vertices is chosen depends only on what happens in the colors of a and b. It does not depend at all on whether any other collection of vertices in the other $n-2$ colors are chosen. This implies the event "a and b are both chosen" is dependent only on those pairs of adjacent vertices which share a color either with a or with b. There are exactly 11 vertices on the cycle sharing a color with a (a with other ten vertices). Each of these vertices is involved with 2 pairs. Hence, there are 21 pairs other than (a, b) which include the same color as a. The same holds for b. As we do not know whether these two sets are disjoint or not (both is possible), therefore, we take $d = 21 + 21 = 42$ in the Lovász Local Lemma. Then we will obtain $e \cdot p \cdot (d+1) < 2.72 \cdot \frac{1}{121} \cdot 43 < \frac{11696}{12100} < 1$. Hence, there is a positive probability that none of the bad events occur. In our case it means, that our set contains no pair of adjacent vertices. This implies that a set satisfying our conditions must exist.

q.e.d.

As an example that there are several versions of the Lovász Local Lemma we show the version named General Lovász Local Lemma or its corollary - the Asymetric Lovász Local Lemma, which allows large differences in the probabilities of the "bad" events and therefore, it is more flexible:

Lemma 3. *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ be a finite set of events in the probability space Ω. For $A \in \mathcal{A}$ let $\Gamma(A)$ denote a subset of $\mathcal{A}$ such that A is independent from the collection of events $\mathcal{A} \setminus (\{A\} \cup \Gamma(A))$. If there exists an assignment of reals $x : \mathcal{A} \to (0; 1)$ to the events such that*

$$\forall A \in \mathcal{A} : P(A) \leq x(A) \cdot \prod_{B \in \Gamma(A)} (1 - x(B))$$

then the probability of avoiding all events in $\mathcal{A}$ is positive, in particular

$$P(\overline{A_1}, \overline{A_2}, \ldots, \overline{A_n}) \geq \prod_{A \in \mathcal{A}} (1 - x(A)).$$

The symmetric version follows immediately from the asymmetric version by setting $x(A) = \frac{1}{d+1}$ for all $A \in \mathcal{A}$ to get the sufficient condition $p \leq \frac{1}{d+1} \cdot \frac{1}{e}$ since $\frac{1}{e} \leq \left(1 - \frac{1}{(d+1)}\right)^d$.
Also, applying Lemma 3 with $x_i = \frac{1}{(d+1)}$ yields Lemma 2 as a special case. Also the proof of Lemma 3 can be obtained by mimicking the proof of Lemma 2.

In the proofs based on all versions of Lovász Local Lemma no method of determining an explicit element of the probability space in which no "bad" event occurs is given. Everytime we are dealing only with probabilistic arguments (proofs are nonconstructive). These are often less clear for students then the arguments that we are using while dealing with constructive proofs. Therefore, students rather try to omit these kinds of proofs, however the Lovász Local Lemma is a powerful tool commonly used in the graph theory research. Thanks to the Lovász Local Lemma we are able to prove the existence of certain complex mathematical objects with a set of prescribed features even if we do not know how it looks like.
A typical proof proceeds in this way: We operate on a complex object in a random manner and then use the Lovász Local Lemma to bound the probability that any of the features is missing. The absence of a feature is then considered to be "bad". When we are able to show that all such bad events can be avoided at the same time with non-zero probability, the existence of the object follows.
The most tricky part is often the determination of dependencies of some events, and hence the calculation of the probability that some event happens. This often requires a bit of arguing, some algebraic calculations and not rarely it is also long. Due to the calculations of probabilities it also sometimes, but not always, gives strange looking results. See for example following theorems that have been orginaly proved via Lovász Local Lemma.

Theorem 4. *Let G be a graph of maximum degree Δ and G' be a graph obtained from G by attaching to each vertex v of G a connected graph H_v of tree-depth at most z, i.e., identify v with some vertex of H_v, then G' has Thue choice number at most $\lceil (2^z - 4)\Delta^{2^z-4} e^{4(2^z-3)(2^z-2)} \rceil$.*

Theorem 4 was proved in [8].

Using the Lovász Local Lemma Grytczuk, Przybyło and Zhu [11] proved that the assignment of lists of length 4 satisfy for creating nonrepetitive list vertex colouring of arbitrary long path:

Theorem 5. *Every path P_n satisfies $\pi_\ell(P_n) \leq 4$.*

This is a very nice result that gives the difference at most one between the Thue choice number and Thue chromatic number of a path[3].

The Lovász Local Lemma was effectively used many times on the Thue type coloring problems - see e.g. [1], [8], [11], [26], [27]. Among all these we mention the following theorem which was proved in [1]:

Theorem 6. *There exists a constant $c > 0$ with the following property: For every integer $\Delta > 1$, there exists a graph G with maximum degree Δ such that every nonrepetitive vertex colouring of G uses at least $c\frac{\Delta^2}{\log \Delta}$ colours.*

[3]The same result with more constructive proof was later published in [10]

As the proof of the above theorem was probabilistic, the big advantage what it gave was that the theorem is also valid for $\pi_\ell(G)$ - the list version parameter. And exactly this advantage is one of the main advantages of probabilistic proofs.

IV. Conclusion

We have shown that both constructive and nonconstructive proofs are worth in graph theory research. It is not possible to say which ones have more advantages. It strongly depends on the concrete graph theory problem.

When we know lot of features of the graph or family of graphs, constructive proofs are really effective. Constructive proofs are usually easy readable and do not require a lot of knowledge from other mathematical disciplines, therefore, they are preffered by students. Disadvantage is that these proofs sometimes require some case-analysis what makes the proofs from time to time too long.

When there is not known too much about the structure of a graph or family of graphs, more apppropriate is to use probabilistic proofs to prove some graph properties. By probabilistic proofs we are dealing with random phenomena. Therefore, some knowledge from probability theory is here of use. In this way we are able to prove the existence of certain complex mathematical objects without knowing how it looks like. Dealing with probabilistic arguments is less clear for students than with the constructive ones. Here were focussed on Lovász Local Lemma based proofs, where the hard part is the determination of dependencies of some events and the calculation of the probability that some event happens is sensitive. The algebraic calculations might be also long and the result might look strange. Moreover, probabilistic approaches often do not give tight bounds.

Somewhere in the middle between above methods is newly developed entropy compression method. But it is still quite technical. It is a method that very suits for graph colouring problems of Thue type and in many cases (but not all) gives better constants as the ones in the results obtained by Lovász Local Lemma (see e.g. [25], [26]). Even if it seems to be a strong tool in future discrete mathematics proofs, it has also its limits.

Therefore, we can only repeat, that getting in touch with as many investigation methods as possible is a big promise of moving the research forward. For the students it is very helpfull when they got the information about some of them already during their study. And when the one graph theory school is not that much familiar with the very method the other one might help. Different knowledge, views and accesses lead to quicker forward in acquiring the results in the respective researched area.

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References

[1] N. Alon, J. Grytczuk, M. Hałuszczak, O. Riordan, *Non-repetitive colourings of graphs*, Random Structures & Algorithms, 21(3-4), (2002), 336–346.
[2] N. Alon, J. H. Spencer, *The probabilistic method*, Wiley-Interscience, New York 2000.
[3] J. Barát, J. Czap, *Facial nonrepetitive vertex coloring of plane graphs*, Journal of Graph Theory, 74(1), (2013), 115–121.
[4] J. A. Bondy, U. S. R. Murty, *Graph theory*, Springer Verlag, 2008.
[5] B. Brešar, J. Grytczuk, S. Klavžar, S. Niwczyk, I. Peterin: *Nonrepetitive colorings of trees*, Discrete Mathematics, 307(2), (2007), 163–172.
[6] P. Erdös, L. Lovász, *Problems and results on 3-chromatic hypergraphs and some related questions*, In: A. Hajnal, R. Rado, V. T. Sós, (Eds.) *Infinite and finite sets (to Paul Erdös on his 60^{th} birthday)*, Proceedings of the Colloquia Mathematica Societatis Janos Bolyai 10, Keszthely, Hungary, 1973, II. Amsterdam, North-Holland, (1975), 609–627.
[7] P. Erdős, J. Spencer, *Lopsided Lovász Local Lemma and latin transversals*, Discrete Applied Mathematics, 30(2), (1991), 151–154.
[8] F. Fiorenzi, P. Ochem, P. Ossona de Mendez and X. Zhu, *Thue choosability of trees*, Discrete Applied Mathematics, 159(17), (2011), 2045–2049.
[9] J. Grytczuk, *Thue type problems of graphs, points and numbers*, Discrete Mathematics, 308(19), (2008), 4419–4429.
[10] J. Grytczuk, J. Kozik, P. Micek, *New approach to nonrepetitive sequences*, Random Structures Algorithms, 42(2), (2013), 214–225.
[11] J. Grytczuk, J. Przybyło, X. Zhu, *Nonrepetitive list colourings of paths*, Random Structures & Algorithms, 38(1-2), (2011), 162–173.
[12] J. Harant, S. Jendroľ, *Facial non-repetitive vertex colouring of some families of 2-connected plane graphs*, IfM Preprint M 11/04, Institut für Mathematik, Technische Universität Ilmenau, Ilmenau, 2011.
[13] J. Harant, S. Jendroľ, *Nonrepetitive vertex colourings of graphs*, Discrete Mathematics, 312(2), (2012), 374–380.
[14] F. Havet, S. Jendroľ, R. Soták, E. Škrabuľáková, *Facial non-repetitive edge-colouring of plane graphs*, Journal of Graph Theory, 66(1), (2010), 162–173.
[15] B. Haeupler, B. Saha, A. Srinivasan, *New constructive aspects of the Lovász Local Lemma*, Journal of the ACM (JACM), 58(6), (2011), #28.
[16] S. Jendroľ, E. Škrabuľáková, *Facial non-repetitive edge-colouring of semiregular polyhedra*, Acta Universitatis Mattia Belii, Ser. Mathematics 15, (2009), 37–52.
[17] B. Keszegh, B. Patkós, X. Zhu, *Nonrepetitive colorings of lexicographic product of graphs*, Discrete Mathematics and Theoretical Computer Science, 16(2), (2014), 97–110.
[18] K. Kolipaka, M. Szegedy, Y. Xu, *A sharper local lemma with improved applications*, Approximation, Randomization, and Combinatorial Optimization: Algorithms and Techniques, Springer Berlin Heidelberg, (2012), 603–614.
[19] J. Kranjc, B. Lužar, M. Mockovčiaková, R. Soták, *On a generalization of Thue sequences*, Electronic Journal of Combinatorics, 22, (2015), #2.
[20] A. Kündgen, M. J. Pelsmajer, *Nonrepetitive colorings of graphs of bounded tree-width*, Discrete Mathematics, 308(19), (2008), 4473–4478.
[21] R. A. Moser, *A constructive proof of the Lovász Local Lemma*, In: Proceedings of the 41^{st} Aannual ACM Symposium on Theory of Computing, (2009), 343–350.
[22] R. Moser, G. Tardos, *A constructive proof of the general Lovász Local Lemma*, Journal of the ACM, 57(2), (2010), 1–15.
[23] W. Pegden, *Highly nonrepetitive sequences: Winning strategies from the local lemma*, Random Structures Algorithms, 38(1-2), (2011), 140–161.
[24] I. Peterin, J. Schreyer, E. Škrabuľáková, A. Taranenko, *A note on the Thue chromatic number of lexicographic products of graphs*, arXiv preprint arXiv:1409.5154, (2014), 1–6.
[25] J. Przybyło, *On the facial Thue choice index via entropy compression*, Journal of Graph Theory, 77(3), (2014), 180–189.
[26] J. Schreyer, E. Škrabuľáková, *On the facial Thue choice index of plane graphs*, Discrete Mathematics, 312(10), (2012), 1713–1721.

[27] J. Schreyer, E. Škrabuľáková, *Total Thue colourings of graphs*, European Journal of Mathematics, 1(1), (2015), 186–197.
[28] E. Škrabuľáková, *The Thue choice number versus the Thue chromatic number of graphs (A Survey)*, manuscript, (2015), 1–19.
[29] J. Spencer, *Asymptotic lower bounds for Ramsey functions*, Discrete Mathematics 20, (1977), 69–76.
[30] A. Thue, *Über unendliche Zeichenreichen*, Norske Videnskabers Selskabs Skrifter, I Mathematisch-Naturwissenschaftliche Klasse, Christiana, 7, (1906), 1-22.

The need and the practice of formative assessment in mathematics at university

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***Abstract*—The formative assessment is the heart of effective teaching. This makes us consider its aspect very thoughtfully and find the ways of its embedding into teaching of mathematics at university level. The core constructs of formative assessment that can be easily implemented into the classes are the goals' setting and communication, the feedback and the questioning. Each of them is discussed in the paper; the discussion characterizes the main features of quality of each of them and supplement it by the author experience. The paper can serve as the source for the formative assessment implementation and for the next research.**

***Index Terms*—Formative Assessment, Effective Teaching, Mathematics**

I. Introduction

Formative assessment is acknowledged by researchers as the assessment *for* learning what is the opposite or complement of assessment *of* learning. Black and Wiliam consider this type of assessment "the heart of effective learning" [1]. This claim, repeatedly reaffirmed by research, should produce enough motivation to remember the students' needs while introducing new methods into education. It proposes us to think of particular students, their specific learning styles, preconceptions and misconceptions. University mathematics education often lacks this feature, maybe it is because of the anonymity of classes where many students are present, or because it can be difficult to implement the formative assessment into the system where is very high pressure on the summative assessment. Whatever reason is behind this situation, we believe that the formative assessment is inevitable part of university mathematics education. New technologies are bringing many opportunities for learning into education, but mathematics educators should not take for granted that they actually improve learning. These new opportunities are still only the opportunities, which have to be utilized to fulfill their potential. The formative assessment is a tool, which helps the teacher and the students keep track of learning, motivate learning and even cause the learning.

It can be understood as a very specific type of communication. This approach allows us to reveal the core constructs of the formative assessment and dynamic relations between them. Those are: goals setting, formal assessment tools, feedback, questioning, peer-assessment and self-assessment. The first four constructs are more teacher-oriented, the last two require students' activity. In this paper, we focus our attention on the teachers' formative assessment, but we explain connections and its importance for the two other parts. This selection is not the denial of "non-teacher-assessment" importance, yet it is the necessary reduction of the topic to be able not just to skim the surface. Besides, good teachers' assessment is the base for the good self and peer assessment what motivates us to support the teachers' assessment at the first place. We recognize following four constructs within it: the goals - their setting and communication, the assessment tools, the feedback provided by a teacher and the questioning. The paper provides detailed characterizations of particular constructs and suggests different options for their implementation into process of education at university level supported by the author's experience. We leave out the formative assessment tools, because our own inexperience in this area, but we take a look at assessment tools as one of the way how to communicate goals.

II. The core constructs of the formative assessment

"We use the general term assessment to refer to all those activities undertaken by teachers - and by their students in assessing themselves - that provide information to be used as feedback to modify teaching and learning activities. Such assessment becomes formative assessment when the evidence is actually used to adapt the teaching to meet student needs." [2] This classical definition by Black and Wiliam reveals the core of the formative assessment. It is the assessment that takes a part during the process of education, it directs this process and moreover it recognizes the progress as something valuable, not only the results. The role of the assessor is assigned to a teacher in the summative assessment. On the contrary, both - a student and a teacher - are supposed to assess the student's progress in formative assessment. Furthermore, not only the particular student assess him/herself, but the other students take the part as well. As a matter of these facts we discern teacher's formative assessment, self-assessment and peer-assessment. There are five key strategies of formative assessment named by Leahy at al [6]. These five strategies are used in all three parts of formative assessment, of course in different way with the respect to the distinct roles of the teacher, the learner and the peers in the process of learning [11].

A. The Goals

Clear targets, according to Chappius [3], support teachers and students role in education process. Teachers:

- know what to teach,
- know what to assess,
- know what instructional activities to plan,
- can avoid "coverage" at the expense of learning,
- have ability to interpret and use assessment results,
- have system for tracking and reporting information and
- have common ground for working collaboratively with other teachers.

Students:

- understand what they are responsible for learning,
- are ready to receive teachers' feedback,
- are prepared to self-assessment,
- are able to set partial goals and track their achievement and
- are able to reflect on and share their own progress.

Benefits of clear targets by Chappius imply, that the learning goals should be well communicated to make students able to take control over their own learning. Students' ability to manage their learning can be possibly discussed at elementary or high school level. Yet, it has to be developed and required at university level, where *adult* people are educated. Students spend a plenty of time with self-study and particularly at study mathematics it is the assumption for the success. It is the time that cannot be supervised by a teacher. If the goals are clear to all the students, those of them who decide to work hard are much more probable to "work hard in the correct direction".

Consequent and very natural question is, how to define and communicate goals for the university mathematics education. Before particular goals are communicated they inevitably have to be defined. The choice of topics covered in mathematics courses should not be the random matter and it should not be the thing of "historical" conventions. Somehow, we presume that our students need all of the stuff we studied, or was the content of the lectures many years ago. Of course, there are different targets for those who are educated as future mathematicians, as mathematics teachers, chemists, physicist or engineers. And of course, it takes a plenty of time to find the compromise between the requirements of the practice and educators' vision about mathematically well educated students.

There are more ways how to communicate the goals to students. A few of them are presented below:

- **Check list** of required knowledge, methods of solutions, definitions, theorems, proofs etc. On one hand, such communication can be quite overwhelming. On the other hand, many students in Slovakia are tempted to delay the study duties in the name of "enough time". Such overwhelming communication can possibly force them to realize the difficulty of mathematics study.
- **Structured assignments** of mathematical tasks in such a way that makes sense in the terms of set goals. The author has an experience where assignments for the freshmen subject "Introduction into mathematics" were structured based on the method of solution. E.g. series on exponential and logarithmic equations was structured as follows:
  Solve the exponential equations in the $\mathbb{R}$.
  - Arrange both sides to the same base; e.g.
    $$5^{x^2-x-4} = 25^{1-x}$$
  - Use the appropriate substitution; e.g.
    $$25^x - 5^x - 6 = 0$$
  - Logarithm both sides of the equation with the logarithm to the appropriate base; e.g.
    $$2^x = 5$$
  - The tasks of different types.

  Solve the logarithmic equations in the $\mathbb{R}$.
  - Use the definition of logarithm; e.g.
    $$\log_2(x^2 + 4x + 3) = 3$$
  - Arrange both sides to the same base; e.g.
    $$\log(x-2) + \log(3+x) = \log 6$$
  - Use the appropriate substitution; e.g.
    $$\log^2 x - \log x^4 = -3$$
  - The tasks of different types, combination of logarithmic and exponential equations included; e.g.
    $$\log_2(9 - 2^x) = 3 - x$$

  Each part contained three or four equations, those of different types about ten of them. This way enhances students' awareness of different types of equations solutions, moreover it enables them to discern whether they cannot solve particular type or they are not able to find out what type of solution leads to success.
- **Rubrics** of assessed abilities, skills or mathematical performance. The difference between check list and the rubrics is in the depth of information. The rubrics contain the possible levels of performance. It is very useful for home assignment.
- **Assessment tools** are implicit way of goals communication. Especially, at university level, where the test assignments are shared and discussed between students, we have to be aware of it. "What you test is what you get." [8] In other words, if any of the set goals is not the part of the official assessment, it is probably not going to be reached. And now, many of mathematics educators are very proud of mathematics being the subject that advance thinking. This is the excuse for integrating many advanced knowledge into mathematics courses at university or even at high schools. Yet, how is thinking

development embedded into the mathematics classes and how is it implemented into assessment? It is important to think about it, because assessment *does* matter for learning.

B. The Feedback

”Feedback is information about the gap between the actual level and the reference level of a system parameter which is used to alter the gap in some way.” [1] Motivation is, based on Hejnys claims, first step in learning process. His definition is as follows: ”Motivation is a tension between existing and desired states.” [7] If motivation is *the tension* between where we are and where we want to be, than *information* about this gap can cause *the tension*. In other words, well performed feedback can generate motivation to learn. First, to perform feedback well, the educator should gain the maximum of information hidden behind students' solutions, answers, and results. To be more specific, the educator should know:

- **if the result is correct or not.** It is usually the first step when reading students solutions to check what the final result is. Even if the assignment is very open-ended. It is important to realize, that correctness of result is good enough information for summative assessment, yet it is very superficial when talking about formative assessment.
- **which of processes are mathematically correct and which are not.** Mathematic teachers are mostly open to assess the process of solution as well. It makes sense, because teachers feel like more objective: It is appropriate to score solutions based on correct steps that student was able to perform. Scoring is the part of the summative assessment and it can be decided by a teacher if the process will be scored or not. There is a statement of one accountant remembering her university studies: ”I keep thinking about my teacher of accounting. She was very strict concerning numerical errors. There was zero score for the assignment if the result was not correct. It was pretty though to pass the exams. Just, you know, we studied harder and now I work as an accountant and it is completely the same - zero tolerance to errors. She knew how to prepare us for this work. I appreciate it.” We are not up to cancel process-based scoring, we are just up to concerning its effect on students learning. On the other hand, the formative assessment requires very deep analyses of the process of solution that result into the next two important information.
- **what is the level of understanding of particular mathematical concept.** In other words, what preconceptions, misconceptions do the students have.
- **what activities should be undertaken by the student to get to the higher level of understanding.** As constructivism claims, the knowledge has to be constructed by the learner and the learner constructs his/her knowledge by activity. The right activities are the most powerful feedback that can be provided by the teacher.

Nyquist [10] in his research review differs following types of feedback:

- Weaker feedback only (effect size: 0,16): students are given only the knowledge of their own score or grade, often described as “knowledge of results”
- Feedback only (effect size: 0,23): students are given their own score or grade, together with either clear goals to work toward or feedback on the correct answers to the questions they attempt, often described as “knowledge of correct results”
- Week formative assessment (effect size: 0,30): students are given information about the correct result, together with some explanation.
- Moderate formative assessment (effect size: 0,33): students are given information about the correct results, some explanation, and some specific suggestion for improvement.
- Strong formative assessment (effect size: 0,51): students are given information about the correct results, some explanation, and specific activities to undertake to improve.

Second, the educator should do everything in such manner, which helps students to receive the feedback. Social psychologists specified a few features of effective feedback. [9] Those are:

- **it is specific, not general.** This is covered by previous section - a teacher has to communicate not only if the result and process is correct, but he/she should provide explanation and suggest activities to improve students learning.
- **it is headed towards behavior not personality.** A teacher does not judge student's personality -clever or stupid, smart or too slow. Conversely, he/she assess only and only the results of the work. The praise or criticism do not help growth.
- **its timing enables enhancement.** It means, it is provided *before* summative assessment, before the final exams. The author experience from Introduction to the mathematics and Geometry classes offers one way how to reach this goal. At the beginning of the class all important knowledge is revised or inquired by the whole group frontally. Next, students (freshman and those at their third or fourth year at university) are told to solve several math problems by themselves. The teacher has enough time to check solutions of the actual students, ask questions and provide feedback. If the assignments are well prepared, misconceptions are revealed in time.
- **it describes the teachers view.** This might seem to be only the matter of formulation and truly, it is. However, it can be greater difference than we think. Let us consider statements below that are focused on students solution of quadratic equation $x^2+4x=0$. Student used discriminant formula and subsequently the formula for the roots of quadratic equation.
  - A: You have not realized that this quadratic can be solved in much easier way.
  - B: It seems to me that you prefer this type of solution. Can you see any other?

The statement A suggests the mistake and it can be easily interpreted as blaming. Such communication restricts creative environment. The statement B provide the student enough space to realize that his/her solution is only one between many and afterwards the student (not the teacher) can state that the first one was not the most effective.

- **it is given on behalf the students need.**
- **it is required by students not forced by teacher.**

C. The Questioning

The questioning is very specific mean of formative assessment. The researchers speak about proximal formative assessment or discourse-based assessment. [4] Just to underpin this fact, let us reconsider the definition of formative assessment again: "We use the general term assessment to refer to all those activities undertaken by teachers - and by their students in assessing themselves - that provide information to be used as feedback to modify teaching and learning activities. Such assessment becomes formative assessment when the evidence is actually used to adapt the teaching to meet student needs." [2] We can see questioning fits this definition from several reasons: first, the questioning is the activity undertaken by teachers and their students; second, it is a rich source of information about particular students, if it is implemented properly; third, if teacher is truly involved in the teaching, he/she makes changes based on the answers that are given to him/her by the students. This prompts us to emphasis the influence of the questioning practice on students learning. We specified a few areas that influence the way of the questioning of a particular teacher. The first of them are teacher beliefs. In one of our previous papers [5] , we have identified core beliefs that cause the differences between teachers classroom questioning practice. Those are: beliefs about talent and effort, beliefs about students' preferences and beliefs about outer conditions. In short, if mathematics teacher believes his/her students are not able to carry out advanced thinking, he/she will teach them just to remember algorithms and will not challenge them with good questions. If mathematics teacher believes his/her students prefer just to sit at classes and do nothing, he/she will hardly ever force them to do activities or if so, he/she will face the confrontation. If mathematics teacher believes that outer conditions (number of students at class, lack of time, etc.) are the strong restriction of a good education, sooner or later he/she can start to use this as an excuse for formalism.

The second area is teachers' preparation for the class. We believe teacher should have prepared not only assignments but at least the principal questions that he/she will ask. It is not the trivial matter to find the accurate question to localize misconception or to demonstrate particular knowledge. Let us give an example: A student is solving quadratic inequality $x^2-6x+10>0$, the discriminant is negative and the student's interpretation is that the empty set is the solution, however this is not the true conclusion. Nevertheless, the student is fine with this and he has no doubts that this is the solution of the problem. This is very common mistake and the teacher has a chance to get ready for such a situation.

- Spontaneous questions: What is wrong with your solution? Is this all right?
- Questions prepared in advance: What does it mean, that the empty set is the solution of the inequality? Can you sketch the situation?

Spontaneous questions are useful mostly when the solution is incorrect and they implicitly contain the information about the present mistake, whereas questions that are more thoughtful can be asked even the solution is correct and let the student figure out incompatibility between the solution and the sketch.

The third area can be very difficult for students to accept and probably it is good to explain the reason in advance to avoid the resistance. Mathematics educator is not the person who has to answer all the students questions. Usually, if a student is able to pose a good question, he/she is smart enough to find the answer as well. That is the reason why teachers should consider very carefully which students' question he/she will answer. The author experience about this are quite encouraging. After student presents his/her question, we suggest to pass it immediately back: "You tell me, if it is correct. I do not know, it is your solution." "You tell me, where is the mistake. You have just found out that something is not working well, so please, try to figure out why." etc. At the beginning, students were a little bit confused, but as time is going, they do not expect the teacher to be the source of "shortcuts" any more.

III. Conclusion

The formative assessment is the powerful tool for learning enhancement and it has to be used to ensure that new approaches in university mathematics education cause the positive effect in students learning. Based on the person who provide assessment, we can distinguish three types of the formative assessment: teacher's assessment, self-assessment and peer-assessment. All three parts are necessary, still the teacher's assessment is the ground for the other two types and that is the reason to go deeper in this, first. The three main constructs of teacher's formative assessment were discussed above: the goals setting and communication, the feedback and the questioning. First, the goal is a clear definition of desired state of students knowledge, skill or ability. Without them a good feedback is impossible, because it is the information about the gap between the actual and referenced state. Moreover, if goals are appropriately communicated, the students are empowered for self and peer assessment. The author's experience with goals communication through explicitly structured assignments seems to support students learning and enhance discussion while consulting students' troubles. Second, the feedback quality relies on the information, which is a teacher able to "read" out of the students' performance and on the way how this gained information is given to the particular student. Effective feedback providing is kind of art that can be practiced and teachers are able to get better at this activity. The author manages her classes in such manner that enables her to provide feedback to particular students immediately while solving the problems. Third, the questioning is the very special way of formative assessment and it is the most usual one. That is the motivation to handle it at very good level as the part of the formative assessment. The questioning can be, based on the author's experience, led in such way that changes students understanding of teacher's role in the process of learning.

The paper can serve as the source of inspiration for the formative assessment implementation at mathematics classes at universities. It provides models supported by the author's experience, which are gained through Introduction to mathematics, Geometry I and Geometry II courses. Furthermore, there is a lot of opportunities for research on formative assessment implemented in mathematics classes at university. We suggest these research questions: How are students' beliefs about mathematics changed by the good formative assessment? How is students mathematical thinking influenced by the good formative assessment? How is students motivation to study mathematics "class by class" affected by the good formative assessment?

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REFERENCES

[1] P. Black, D. Wiliam, *Assessment and classroom learning*, Assessment in Education: Principles, Policy and Pracice, 5(1), (2006), 7–74.

[2] P. Black, D. Wiliam, *Inside the black box: Raising standards through classroom assessment*, Phi Delta Kappan, 80(2), (1998), 139–148.

[3] J. Chappius, R. Stinggins, S. Chappius, J. Arter, *Classroom assessment for student learning: Doing it right, using it well*, (2nd Ed), Upper Saddle River, Pearson Education, Inc., New Jersey, 2012.

[4] C. H. Chen, M. D. Crockett, T. Namikawa, J. Zilimu, S. H. Lee, *Eighth grade mathematics teachers' formative assessment practices in SES-different classrooms: a Taiwan study*, International Journal of Science and Mathematics Education, 10(3), (2012), 553–579.

[5] V. Hubeňáková, D. Šveda, *Beliefs of teachers of mathematics about teaching in a constructivst way*, Proceedings of the 10th International Conference on Hands-on Science, M. F. Costa, B. V. Dorrío, M. Kireš, (Eds.), 2013, 1-5 July, Pavol Jozef Šafárik University, Košice, Slovakia, (2013), 36–41.

[6] S. Leahy, C. Lyon, M. Thompson, D. Wiliam, *Classroom assessment: minute-by-minute and day-by-day*, Educational Leadership, 63(3), (2005), 18–24.

[7] N. Stehlíková, *Náměty na podnětné vyučování v matematice*, Pedagogická fakulta, Univerzita Karlova v Praze, Praha, 2007.

[8] M. Swan, H. Burkhardt, *A Designer Speaks: Designing Assessment of Performance in Mathematics*, Educational Designer, 2(5), (2012), #19.

[9] R. Thomson, C. Mabey, *Developing human resources*, Butterworth-Heinemann Ltd., Oxford, 1994.

[10] D. Wiliam, *The role of formative assessment in effective learning environments*, In: H. Dumont, D. Istance, F. Benavides, (Eds.) *The Nature of Learning: Using research to inspire practice*, OECD Publishing, Paris, 2010.

[11] D. Wiliam, *Embedded formative assessment*, Solution Tree Press, Bloomington, 2011.

A game on exact mathematical verbalization

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***Abstract*—There are many terms in mathematics that students understand only intuitively. A lot of reasons lead students to use them in a not correct way. Either students are too lazy to use the exact mathematical formulation or they do not know it at all, the effect is the same - mistake. Here we show how one can built the exact mathematical verbalization of students via game. In our example the game is situated to geometry but similar games one can play in other branches of mathematics.**

***Index Terms*—Mathematical Language, Definition, Geometry, Teaching via Game**

I. Introduction

The questions what and how to educate students is the one that a lot of teachers ask (see e.g. [1], [2], [4]). Teachers of mathematics starting from those at primary schools and ending at university level are dealing with a lot of problems connected with education. But at least one of these problems is remarkable at all levels and types of schools and universities: The prolem of not exact mathematical formulations. Students, who read the term x^2 as “x two” often see the difference between $2x$ and x^2, but they are too lazy to read it correctly and do not thing about causalities, or they are not enough patient. We are not rarely getting in touch with this and similar examples in algebra and mathematical analysis. There are also examples in geometry where students are mixing the terms *(stright) line* and *curve*; *circular line (circle)* and *disk*; *point* and *vector*; and many others.

There are many approaches how to force students using exact mathematical language. It is never easy to find the right way to do it. In this paper we choose one among all and describe it on an example from geometry. Our approach is based on a game. Therefore, we consider it to be less stressfull for students like the classical approaches. Moreover, students and teachers are switching their roles in this game what effects higher motivation of students to find the right verbalisation tools.

II. Motivation

It is not easy to motivate the students of secondary grammar schools, high schools or university students to do something voluntary, without pressure and at the same time effective. Therefore, some motivation story might be of use in these situations. A teacher may pretend that he is a new student from foreign country (even foreign planet should be funny and acceptable from the student's side) and that he has big language problems. He has only some basics of the local language and he wants to study geometry here. In the basic language course (he needed to complete before comming here for study) he hardly, but successfully, understood the terms *point; set of points; distance between two points; equivalence; being a member of a set; equivalence of sets; non-equality between two numbers; join; meet* as well as the meaning of the *basic aritmetical operations* and *logical connecting symbols*. Now he wants his schoolmates to explain him some new terms.

III. The Game

Teacher is asking the students (in the game his co-students) to explain him some new terms from geometry. He is careful to stop the students every time, when they use some new term, he does not know. In this case he is asking for explanation. When the given definition of a new term is vauge, he pretend for a small moment that he understand and then try to give an example. As the formulation was vauge, this example has to fulfill it, but should not be an example of the object required in the definition. The teacher is not satisfied with the definition or explanation of the new term unless it is really correct and based on the term *distance*.

It is always good to start with some simple terms as:

- *Line segment* AB with end points A and B - the set of all points X such that $\mathrm{dist}(A,X) + \mathrm{dist}(X,B) = \mathrm{dist}(A,B)$.
- *Half-line* $\overrightarrow{AB}$ - the set of all points X such that $\mathrm{dist}(A,X) + \mathrm{dist}(X,B) = \mathrm{dist}(A,B)$ or $\mathrm{dist}(A,B) + \mathrm{dist}(X,B) = \mathrm{dist}(A,X)$.
- *Line* $\overline{AB}$ - the set of all points X such that $\mathrm{dist}(A,X) + \mathrm{dist}(X,B) = \mathrm{dist}(A,B)$ or $\mathrm{dist}(A,B) + \mathrm{dist}(X,B) = \mathrm{dist}(A,X)$ or $\mathrm{dist}(A,B) + \mathrm{dist}(X,A) = \mathrm{dist}(B,X)$. (In this point there are different ways how to reach the goal, for example we can also define the line $\overline{AB}$ as $\overline{AB} = \overrightarrow{AB} \cup \overrightarrow{BA}$.
- *Circle* C with middle point S and radius r - the set of all points X such that $\mathrm{dist}(S,X) = r$.
- *Disk* K with middle point S and radius r - the set of all points X such that $\mathrm{dist}(S,X) \leq r$.

As soon as there are some terms newly defined, we can use them in new definitions. We can continue and ask for more complicated definitions of terms: *outer ring; triangle; ellipse; hyperbole; parabola;...*
We can also ask for explanation of some properties of objects,

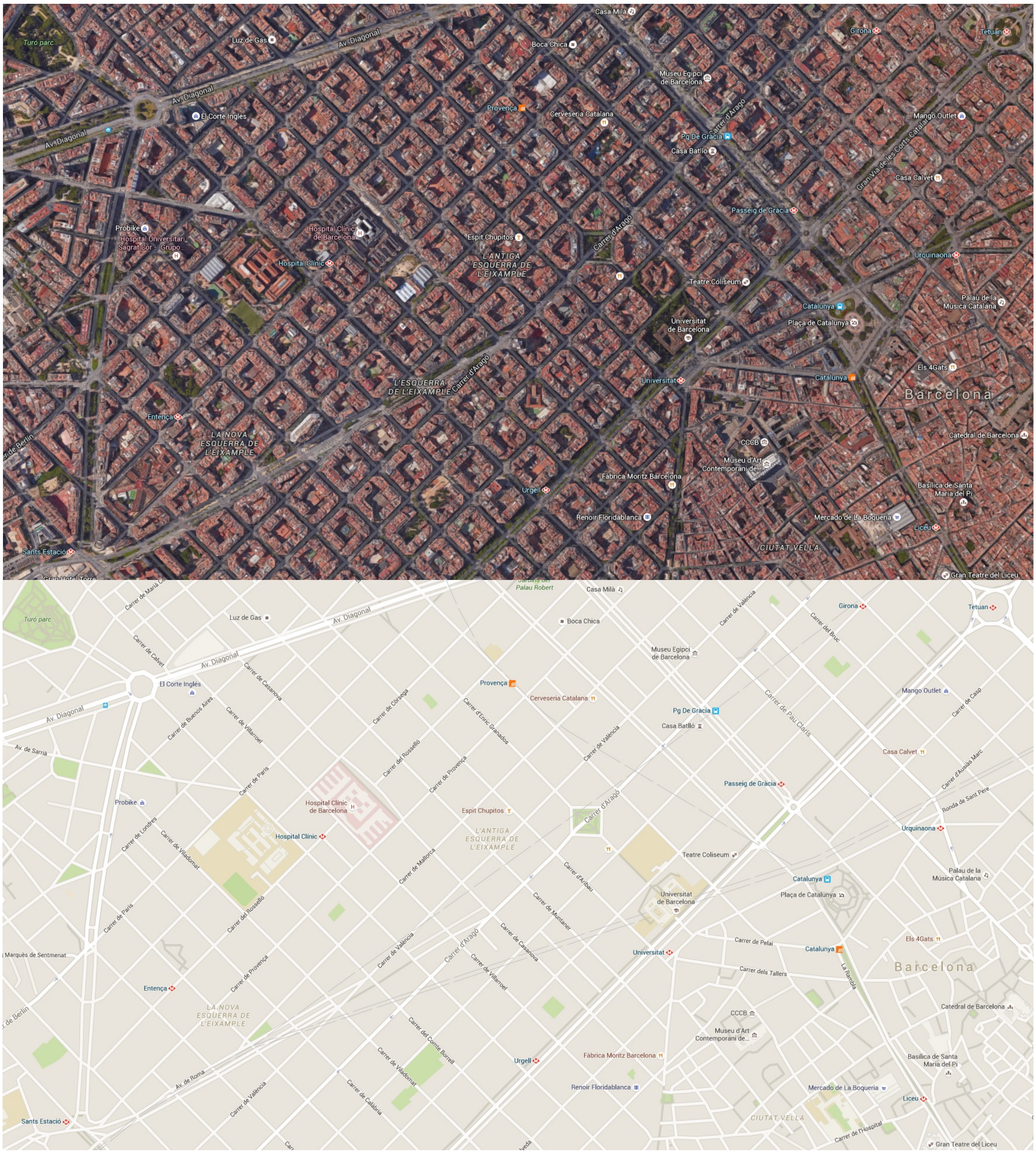

Fig. 1. The street-view and vector map of a part of Barcelona (Spain), [8]

for example describe when two lines p and q are *parallel* (iff all the points X of p have the same distance from q, where $\operatorname{dist}(X, q) = \min\{\operatorname{dist}(P, Q); P \in p,\ Q \in q\}$).
Having this definition we may ask for the definition of *square; rectangle; trapezoid;...* asking when the two lines p and q are *ortogonal* and so on.

Sometimes students tempt the teacher to accept their not preceise definition arguing that the teacher has to understand it, because “it is intuitive”, “he knows what they mean” or “it is not possible that he has so bad language knowledge” and suggest to use some online translator (or similar tool) in order to translate the term from local language to the language of stranger. In this case it is usefull to know and say some sentences in (for students) hardly identificable language (e. g. Latin, Norwegian, Hindi, ...). Afterwards, in most of the cases,

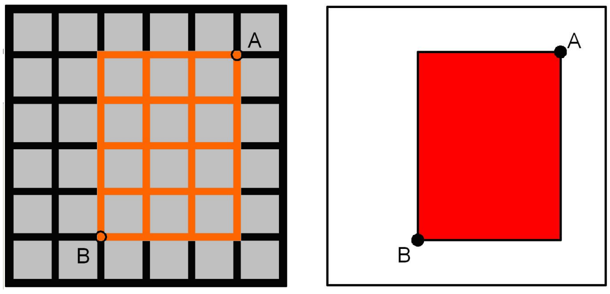

Fig. 2. A line-segment in “postman metrics”

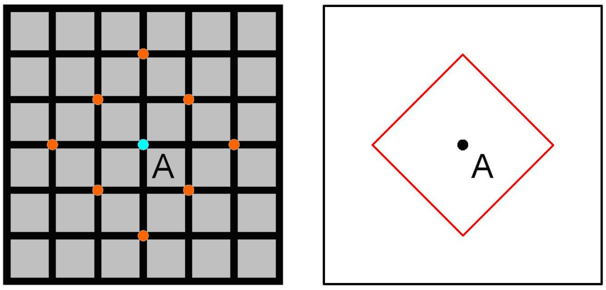

Fig. 3. A circle in “postman metrics”

they again try to focus on better mathematical formulations in the local language.

IV. Reasoning

At the end of the game the teacher should initiate a discussion about inevitability of exact mathematical verbalization and show the students some examples from non-Euklidean geometry.

For example the distance for a postman in cities where the roads are organised more-less like a square grid is not the smallest distance between two points on the map of the city, but the length of the way he needs to walk using the streets - see for example map of Barcelona on Figure 1 or maps of cities in USA - Austin (Texas), New Yourk (New York), Fort Lauderdale (Forida) and so on.

Here the teacher, still playing the foreign student, can repeat some of the definitions he learned from the students and draw pictures in this “postman metrics” [3]. The pictures looks interesting - see for example the drowings of line-segment, disk and circle on Figure 2 - 4.

Similary, we can define some other metrics. In “horse metrics” the distance between two points at infinite chessboard - see Figure 5, is the minimum chessboard L-horse-moves in order to reach the second point from the first one.

The situation here, therefore, looks even more interresting. Again the teacher should repeat some of the definitions of mathematical objects which he got from his students. And again he can draw pictures of the same objects in this “horse metrics” - see for example a drowing of a circle of radius 1

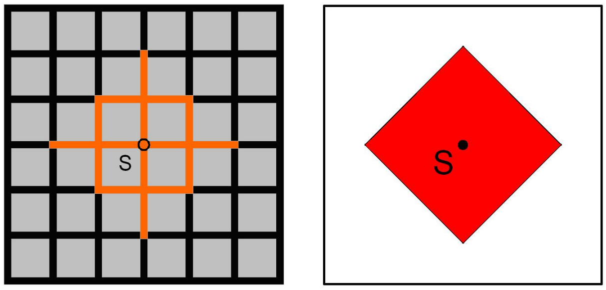

Fig. 4. A disk in “postman metrics”

Fig. 5. Infinite chessboard, [7]

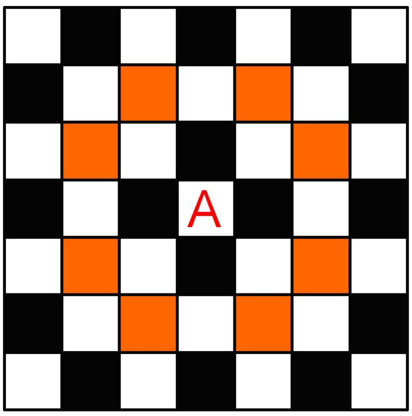

Fig. 6. A circle of radius 1 in “horse metrics”

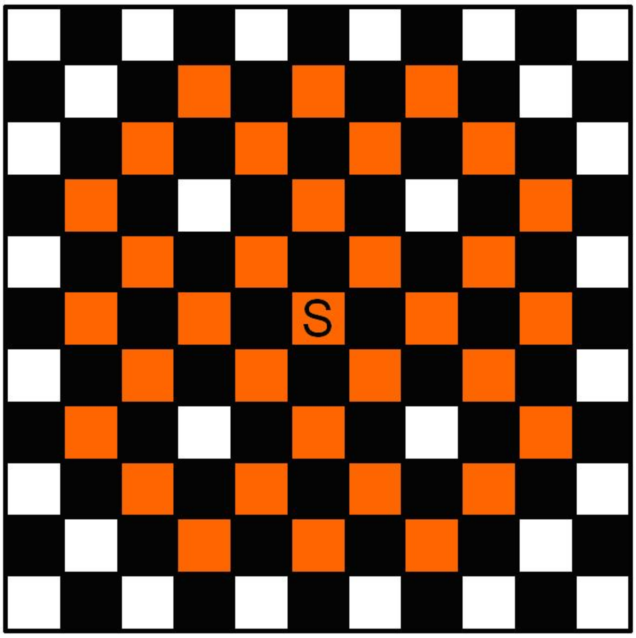

Fig. 7. A circle of radius 2 in “horse metrics”

on Figure 6 and drowing of a circle of radius 2 on Figure 7.

Beside postman and horse metrics the teacher should also announce some other metrics as radial metrics, river metrics and their generalisations, see e. g. [6], and demonstrate how the same geometrical objects look like in these metrices.

V. Conclusion

In this paper we showed how one can built the exact mathematical verbalization of students via game. Our example of the game was situated to geometry. Variations of the above game were used several times at mathematics camps [5] and at motivation seminars for students (or future students) of mathematics. The response was good. Similar games one can also play in other branches of mathematics. Therefore, we consider this game and its variations being worth for students in order to built their awareness of necessity of exact mathematical verbalization.

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References

[1] M. C. Cañadas, M. Molina, S. Gallardo, M. J. Martinez-Santaolalla, M. Peñas: *Let’s teach geometry*, Mathematics Teaching, 218, (2010), 32–37.
[2] H. H. Chipman: *When I teach geometry*, The Mathematics Teacher, 53(2), (1960), 140–142.
[3] L. A. Fernández, V. Martin-Mayor, G. Parisi, B. Seoane: *Spin glasses on the hypercube*, Physical Review B, 81(13), (2010), 134403.
[4] J. D. Klein: *Comment: teaching and learning limits in engineering education*, Engineering Studies, (2015), 1–3.
[5] Kolektív, *Geometrie netradičních metrik* (Geomery of not traditional metrics), In: Program matematických krúžkov, táborov a sústredení, manuscript, 30–35.
[6] Á. Száz: *A common generalization of the postman, radial, and river metrics*, Rostock. Math. Kolloq., 67, (2012), 89–125.
[7] http://www.amazon.com/Cubist-arabian-Waterproof-Printed-sticker/dp/B00SXJ26Z2
[8] https://google.com/maps/

"Event" - the building block of statistics and "argumentation" - the stumbling block of students

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***Abstract*—This paper consists of investigation into students understanding of statistical concepts which they have learned at secondary school. Our research is based on observation of the basic skills like argumentation by using the rudimentary knowledge of the concept mean. We set two word problems aimed at the understanding of the term mean at different levels of difficulty. In this paper - the case study - we analyse students solutions of these problems.**

***Index Terms*—Argumentation, Statistical Concept, Mean**

I. Introduction

Nowadays, the ability to right orient the amount of information that must be correctly handled, has become considerable important because everyday life is accompanied by the ability to access critical data. For that reason the statistical literacy, which becomes the backbone of economic improvement plays an important role in mathematical literacy. Our research interests include understanding how statistics become a useful tool for future professionals (first grade students of material science) and how their statistical learning can be effectively supported. We concentrate on school course Numerical and statistical methods. It consists of lecture (90 minutes in length once a week) and seminar (90 minutes in length once a week). The areas covered in the statistical part of the course are:

1) Probability,
2) Descriptive statistics,
3) Random variables (discrete),
4) Random variables (continuous),
5) Point and Interval estimation. Hypothesis testing (parametric),
6) Hypothesis testing (non-parametric),
7) Correlation analysis. Regression analysis.

We have an intention of improving the teaching process. For this purpose we use methodology Design Experiments [1]. This is based on a manner of successive iterations, testing and revision of teaching process which is constantly improving. So we need to find the efficiency of the present course out. Afterwards, we can design a new course and find the efficiency of the new course out. This cycle is the never-ending battle for teachers. In this article we mention only the first step: the efficiency of the present course. This is able to be realized by the method of pretest-posttest [2].

II. Methodology

We prepared a pretest and posttest for first-year students at the Technical University. The experiment was aiming to find the profundity of the understanding of the basic statistical terms out. For this reason, we employed word problems for the most part because routine algorithmic tasks don't need to detect a formal students' knowledge. The pretest and posttest problems are focused on basic statistical terms: mean, statistical event, base of percentage, diagram and probability. In this article we analyze only one problem from pretest and one problem from posttest, both of them are focused on the terms mean and statistical event.
39 students of technical fields of study had been assigned the pretest (before the first lesson) and the posttest (after the final lesson). They had 30 minutes to work them out. Each problem is analysed separately.

III. Pretest problem

A. assignment

Maximum point score from a test is 30 points. A student passed the test if he/she got at least 16 points in the test.

A) In the first class 12 students took the test. Their point scores are: 25, 5, 0, 13, 30, 16, 0, 9, 3, 19, 25, 5. Calculate an average point score in the class.
B) In the second class 9 students took the test. We know nobody got the full score and two students got the zero score. The average point score in the second class was 13 points. Fill in the table:

Statement	Possible / impossible / certain	Example/ reason
Exactly three students ***failed*** the test.	possible	**0**, **0**, **5**, 16, 20, 23, 16, 17, 20
At least one student ***passed*** the test.		
Everybody ***failed*** the test.		

B. analysis

By this problem we observe how students understand the term mean at two levels. In the first part of problem we observe the basic understanding the mean and application of formula

for calculating mean.
It is necessary to use only formula for calculating mean: the sum of points/the number of students. The manual use of the basic formula is on the reproduction cluster.

$$\frac{25+5+0+13+30+16+0+9+3+19+25+5}{12} = 12.5.$$

The average number of points in group of 12 students is 12.5. In the second part of problem we observe the deeper understanding the formula in not so ordinary situation (connection cluster). Student needs to know how to connect terms like arithmetic mean and sampling frame with terms possible, impossible and certain (reflection cluster). We observe the students' ability to work with the problem as a whole. It is closely related to critical reading and critical thinking. Critical reading is a technique for discovering information and ideas within a text [3]. This part of solution is on the connection level, because of the need for student to work with the formula from the inverse side (unusual method for them).
It is appropriate for solver in the second part of problem to write down the following notes of assignment:

- The average of 9 students is 13 points.
- Two students got 0 points.
- Nobody got 30 points.
- Student passed if he/she got 16 points.

It is possible to determine from the backward application of formula for arithmetic mean that 9 students got 117 points together. As two of them got the zero score, only seven students got 117 points together. So their average is 16.7 points for each (117/7=16.7). This argument should be enough to make up the for example the sampling frame: 0, 0, 16, 16, 17, 17, 17, 17. More appropriate sampling frame to prove an event that is certain to occur is 0, 0, 15, 15, 15, 15, 15, 15, 27.
If the statement "Everybody failed the test" is true, students got no more than 15 points. So maximal total amount of points is 105 ($2 \cdot 0 + 7 \cdot 15 = 105$). It is in direct contradiction to the average from the assignment ($9 \cdot 13 = 117$). It is clear that the statement "Everybody failed the test" is impossible (reductio ad absurdum). So at least one student had to get at least 16 points. Student had to become aware of reasons for impracticability of situation. So argumentative part of solution is on the reflection cluster.
Statements "At least one student passed the test" and "Everybody failed the test" are opposite so realizing this fact helps student to solve the problem.

C. solution

The first part of problem lives up to our expectations. It was really successful because everyone answered correctly.

The biggest obstacle in the second part of pretest problem is the fact that students do not know to notice and keep in mind the whole context of assignment. Frequent errors are:

- Neglect of mean,
- Pretermission the information that two students got the zero score,
- Solvers do not take into consideration that nobody got 30 points,
- Solvers deal with number of students different from 9.

In the case that more than one mistake is in one solution, this is classified as wrong solution. As nobody wrote down the full notation of the assignment (mentioned above) and only very few solvers wrote down an incomplete notation of the assignment (fig. 3) we assume that solvers after a short while put out of mind the input conditions (for example the information about the minimum points for passing the test).

Only two solutions (fig. 1 and fig. 2) are correct (despite the fact that 25 students categorize this event as certain).

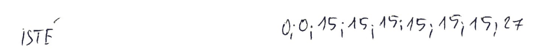

Fig. 1. Correct solution without words.

Solver categorized the event as certain and his/her sampling frame is proving enough so no word is necessary.

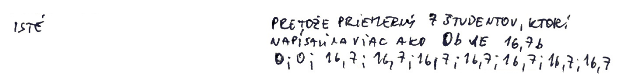

Fig. 2. Correct solution through mean.

Solver categorized the event as certain.

Certain because of the average of 7 students which passed for more than 0 is 16.7.

Four solutions are almost correct (it is possible from the solution read the idea which is partly correct), for example solution (fig. 1). Solver categorized the event as possible incorrectly. The idea of his/her solution is correct through the mean but he/she used the full score in the sampling frame which is forbidden by assignment. In case he/she uses 16 (minimum score for passing the test) instead of 30, the mean should be $11.\overline{7} < 13$ therefore at least one student passed the test.

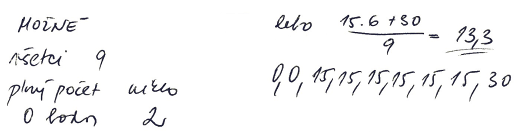

Fig. 3. Solution with incomplete idea.

Solver categorized the event as possible and wrote down the incomplete notation of the assignment (he/she failed to note the mean).

Possible because …

all: 9

the full score: nobody

0 score: 2

Unlike the mean from the assignment (the missing part of his/her notation) the mean of his/her sampling frame is 13.3.

The another common fault is the proof of certain event through non-specific example (6 solutions similar to fig. 4).

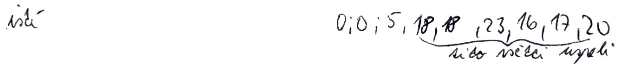

Fig. 4. Proof of certain event through non-specific example.

Solver categorized the event as certain.

all of these passed the test

The another common fault is the proof of possible event through non-specific example (6 solutions similar to fig. 5).

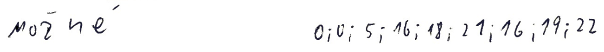

Fig. 5. Proof of possible event through non-specific example.

In the third part of the pretest problem almost everyone categorized the event as impossible (34) but only nine of them were capable of clearly proving (fig. 6).

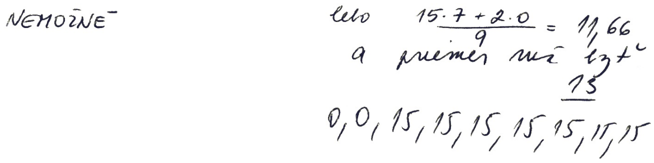

Fig. 6. Correct solution through mean.

IMPOSSIBLE due to $\frac{15\cdot 7+2\cdot 0}{9} = 11.66$ *and mean should be 13*

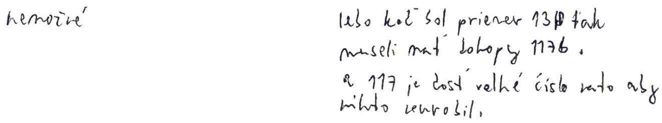

Fig. 7. Funny conception of a mathematical proof.

Impossible because when the mean was 13 so they must have 117 points together and 117 is quite a big number to nobody passed.

IV. Posttest problem

A. assignment

Pupils of one primary school take part in picking paper up every year. School administration guarantee that 12 the most successful pupils will take a trip to Tatralandia in order to increase the pupils motivation.

At a primary part of school are pupils from 1. to 5. grades (children aged 7 to 11) and at an elementary part of school are pupils from 6. to 9. grades (children aged 12 to 15). We know the average age of pupils which took the trip is 10 years.

Fill in the table according to the following conditions:

Condition	Give an example of pupils ages which took the trip./ Give reasons.
Half the pupils were from the primary part of school.	
At least one pupil from each grade took the trip.	
Only pupils from 4, 5, 6 and 7 grades took the trip.	

Footnote: Let's assume that the pupils from one grade are the same age at the end of school year.

B. analysis

This problem is analogous with the pretest problem. The success rate of the first part of the pretest problem shows that we dont need to test the understanding of the term mean at the basic level. So we fix our attention on the understanding of the term mean on the connection and reflection cluster. This problem is simplified by leaving out the terms possible, impossible, certain. The posttest problem is focused only on the making up the sampling frame complying with the given conditions.

It is appropriate for solver to write down the following notes of assignment:

- The average age of 12 pupils is 10 years.

grade	1.	2.	3.	4.	5.	6.	7.	8.	9.
age	7	8	9	10	11	12	13	14	15

a) One from many possibilities is for example 7, 7, 7, 7, 7, 7, 13, 13, 13, 13, 13, 13. This event is only possible.
b) The only example meeting the requirements is 7, 8, 9, 10, 11, 12, 13, 14, 15, 7, 7, 7 so this event is only possible. This part of problem helps us to find out the students ability to make up the suitable sampling frame. We made this problem simple to single out solvers which cant make the basic sampling frame.
c) The example for minimal average age suiting the condition that at least one pupil from mentioned grades took the trip is: 10, 11, 12, 13, 10, 10, 10, 10, 10, 10, 10, 10. The mean is 10.5 therefore this event is impossible.

C. solution

As it is not the question, event classification (possible, impossible, certain) is not the necessary part of correct answer. Nine solutions are correct (for example fig. 8).

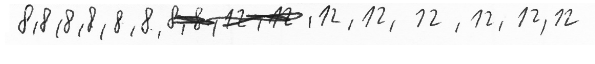

Fig. 8. Correct solution.

There are no responses from eight students and 16 solvers broke the conditions (the number of pupils, the mean or the half).

The second question is more successful, there is 15 correct answers (for example fig. 9).

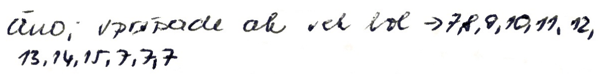

Fig. 9. The only correct solution.

yes, in case ages were 7, 8, 9, 10, 11, 12, 13, 14, 15, 7, 7, 7

There are no responses from eight students as well and 15 solvers gave examples which do not suit the conditions, mostly the mean (for example fig. 10).

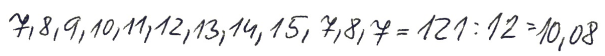

Fig. 10. Example breaks given conditions.

The solver came to realize the broken condition - the mean, but did nothing.

The third part is really interesting from the students solutions side. From our point of view this problem is closed. The starting situation (the given conditions) and the goal situation are closed as well, exactly defined [4]. The only correct solution is the statement *the event is impossible*. But this problem aroused debate into students solution. The first thing that comes into solvers mind is *there must to be at least one possibility* like small children approach. This attitude should disappeared gradually by growing up.

Only four solutions are correct (without the supplemented controversy between the impossible event and an urge to have real solution) (for example fig. 11).

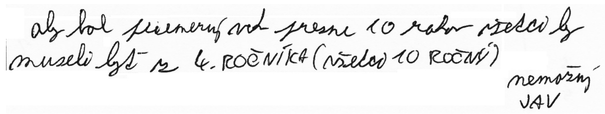

Fig. 11. Correct solution.

everyone should be in the fourth grade to have an average age exactly 10 years (everyone 10 years old), impossible event

In the case the given conditions from the assignment is not closed, exactly defined (not this problem from our point of view), the only correct solution has to have the consideration (fig. 12).

11 solutions show the real possibility *only 12 fourth-year pupils will take the trip* (fig. 13).

For some of solvers the no answer is not the solution. They are trying desperately for finding out the way it should by real (fig. 14).

Or they better not keep the given conditions (fig. 15)

The successful rate is not appreciative and furthermore 11 solvers didnt write down anything.

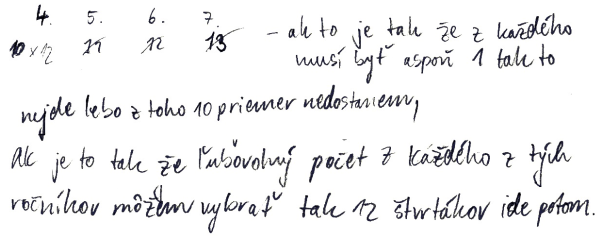

Fig. 12. The only correct comprehensive solution.

if at least one pupil from each grade has to be there, it is not possible, because the mean 10 is not fulfilled, if I can choose an arbitrary number of pupils from each grade then 12 fourth-year pupils will go.

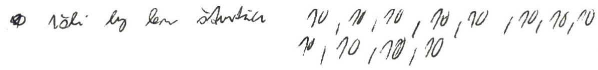

Fig. 13. Solution only consists fourth grade pupils.

only fourth grade pupils will take the trip

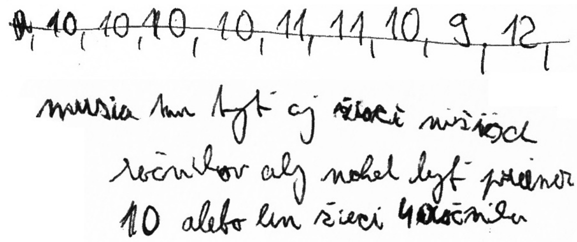

Fig. 14. Bargain for existence of real solution.

there must be pupils from higher grades too to have the mean 10 or only fourth grade pupils

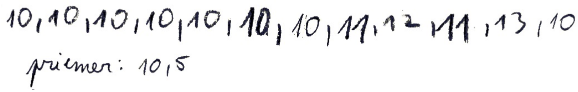

Fig. 15. Example breaks given conditions.

Solver came to realize the wrong mean but do nothing.

the mean is 10.5

V. Conclusion

We started to do pretest-posttest method to improve our teaching. We used to want to compare students skills like argumentation and the influence of the present teaching process on their skills.

We concentrated on the argumentation skill. It is part of critical thinking and we suppose it is the most important thing to know to think. Thinking is more applicable than memorized formula. But skills like argumentation is not capable to be evaluated.

Therefore we have obstacle to evaluate our teaching process. As long as the areas covered in the course are so extensive for so short time we can not concentrate on improving thinking skills.

Acknowledgment

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References

[1] L. T. M. Blessing, A. Chakrabarti, *DRM, a Design Research Methodology*, Springer, 2009.

[2] D. M. Dimitrov, P. D. Jr. Rumrill, *Pretest-posttest designs and measurement of change*, Work, 20(2), (2003), 159–165.

[3] S. Barnet, H. Bedau, *Critical thinking, reading, and writing*, Bedford/St. Martin's, 2013.

[4] E. Pehkonen, (Ed.), *Use of open ended problems in mathematics classroom.*, Department of Teacher Education, University of Helsinki, Hakapaino, Helsinki, 1997.

The reductive argumentation in the school teaching of mathematics. Application of the atomic method

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***Abstract*—This article is a description of a certain didactic experience. this study was the fragmental diagnosis of the ability to use reductive thinking in command of the secondary school students (17-18 years of age). The article also provides references to didactic literature, referring to the proof in mathematics, and methods used, as a way to justify general statements.**

***Index Terms*—Reductive Thinking, Method of the Atomic Analysis, Mathematics**

I. Introduction

Mathematical didactics in Poland has some autonomous problems among other acknowledged scientific disciplines. Methodology of mathematical didactics has the same kind of problems. That is why the scientific research generally makes use of pedagogical methods. However, mathematical didactics differs from pedagogy in terms of the use of this scientific research. The results of the scientific research do not depend on the name of the research or on being a part of a field of knowledge but they depend on a man who makes use of the method (especially on their predisposition, sensitiveness and scientific experience). It is really apparent especially in a following situation: if a mathematician defines at the beginning of the research an object, he will watch over the characteristics deciding about belonging of the object to the definition. Pedagogical definitions are often underspecified, imprecise, they do not include certain conditions.

Mathematical didactics has been trying to work out its own methodological research for centuries. Mathematical didactics draws upon pedagogy, one of the directives which it has required and used is the division of scientific research into a quantitative method and a qualitative method. Quantitative research has started to be used more often since the 60's of the twentieth century ([11]). A. Żeromska (2003) The most characteristic features of this research are:
- precisely defined variables resulting from the set hypothesis, which are measured in the process of the research, and the object of the research,
- the research tools are planned and standardized,
- the organization of the process of the research takes into consideration the needs of statistical methods: random sampling, tools categorization, coding, statistical procedures of data analysis, quantitative description of the results ([6]).

One may enumerate many advantages of the quantitative research such as the possibility to compare the examined phenomenon, establish the strength of the relations, and define the significance of differences between the variables. The disadvantage of the research may include inadequacy of many research problems which require more in - depth approach, for example, intentions, motivations, values, convictions. According to A. Żeromska (2003) a characteristic feature of the qualitative research in contrast to the quantitative one is the holistic attitude to the problem which is oriented towards understanding of the situation in the same way as the person subject to the research. The researchers are oriented towards processes, sequences of events rather than their results. At the beginning there is no hypothesis. The main aim of the research is not to verify the hypothesis but to discover the idea which will allow us to look into the phenomenon ([4]). Data is gathered in an open way with the use of unstructured tools. The possibility to understand the unknown phenomena which were hidden beyond the field of scientific interests, the possibility to reach a great number of details of various complex phenomena are the most important advantages of qualitative research. Qualitative methods do not impose the "language" of the answers, they make it possible to get to know a certain phenomenon in a dynamic and developmental way, *they allow us to learn about a greater context of phenomena which are of some interest to the researcher and identify them in natural conditions* ([5], p. 20). The main disadvantage is the lack of possibility to carry out the research once again - so as to check their descriptions, formulated conclusions, low level of methodological accuracy, descriptive interpretation of the phenomenon, *problems with analyzing of poorly categorized data, poor basis to formulate generalizations and uncertainty of generalized conclusions* ([7], p. 68).

As it has already been stated, mathematical didactics has been pursuing to establish its own research methods. *Method of the atomic analysis* is the example of it, this is one of the methods that belongs to the group of qualitative methods. It was created at the Bratislava Seminar of Mathematical Education. The atomic analysis is based on two ideas: atomization of the process of solution, and the comparative analysis. The main aim of the method is to examine student's thinking processes thanks to sorting out sequences of written ideas and the analysis of thinking progress during the creation of this sequence and the determinant of the process. Symptoms of personal and cognitive behaviours which are used to describe this thinking process are of great significance for the analysis of the process. Students' written works are analyzed paying special attention to one or many symptoms, "atom by atom" ("atom" is a small meaningful part of the solution) ([9]). Method of the atomic analysis is a new idea in the mathematical didactics.

The main aim of this article is the description of the research and its results with the use of the atomic analysis which was introduced to present the fragmentary diagnosis concerning students' use of reductive thinking while proving general tasks.

Aristotle in IV BC introduced the basis for the reductive theory. The basic difference between deduction and reduction is that in case of the deduction on the basis of a conditional sentence and its predecessor one may come to a conclusion of its apodosis. However, in case of the reduction - it is the other way round, on the basis of a conditional sentence and its apodosis one may come to a conclusion of its predecessor ([1]). We choose the premise to the thesis of a proven theorem in a reductive proof. The thesis would be written using the results of the proven theory and would contain results based on definitions, proven theorems and axioms. After finding this premise, we are looking for another premise which the first one would result from. We lead the argumentation so long as we find the premise which is known and obvious. This way of proving the theorem relays on searching for conditions which will be sufficient for the proven thesis and we prove that these conditions are the assumptions of a theorem, previous theorems and axioms. For a given theorem in a form $\alpha \Rightarrow \beta$, we come from a proven theorem β and we look for such expressions w_1, w_2, ...,w_n that there are implications: $(\beta \Leftarrow w_1) \wedge (w_1 \Leftarrow w_2) \wedge ... \wedge (w_n \Leftarrow \alpha)$. The last implication does not have to be there: it is enough that the w_n is the theorem which had been proved before. This scheme may be presented in a shorter way: $\beta \Leftarrow w_1 \Leftarrow w_2 \Leftarrow ... \Leftarrow w_n \Leftarrow \alpha$ ([3]).

Argumentation using reduction may serve to prove porism. However, it is vital to remember that implications in the above description should be possible to be replaced with equipotency.

Argumentation using reduction is rarely used in school teaching in an open way. It might be assumed that this way of teaching is considered to be worse or defective, it happens that in course books, for instance, the content of the proof, which is conducted with the use of the reductive method, is presented in a deductive way. One may, for example, find in a course book the following theorem:

Prove that if $a, b \geq 0$, *then* $\frac{a+b}{2} \geq \sqrt{a \cdot b}$.

It is natural that each student will probably follow this way of dealing with the task:

$$\frac{a+b}{2} \geq \sqrt{a \cdot b} / \cdot 2,$$

$$a + b \geq 2\sqrt{a \cdot b} / ()^2,$$

$$(a+b)^2 \geq 4ab,$$

$$a^2 + 2ab + b^2 \geq 4ab,$$

$$a^2 - 2ab + b^2 \geq 0,$$

$$(a-b)^2 \geq 0.$$

This proof could finish the conclusion: "square of each number is not negative thus initial inequality is real."

In contrast, in the mentioned course book, the text that a student is supposed to deal with is as follows:

It is obvious that for arbitrary $a, b \geq 0$

$$(a-b)^2 \geq 0,$$

$$a^2 - 2ab + b^2 \geq 0,$$

$$a^2 + 2ab - 4ab + b^2 \geq 0,$$

$$a^2 + 2ab + b^2 \geq 4ab,$$

$$(a+b)^2 \geq 4ab / \surd.$$

Because $a, b \geq 0$, *one may write the odds:*

$$a + b \geq 2\sqrt{a \cdot b} / : 2.$$

Finally,

$$\frac{a+b}{2} \geq \sqrt{a \cdot b}.$$

So the theorem is true.

It is possible that at the beginning the author of the proof was using the reductive argumentation but later on because of the unknown reason he presented a student with a deductive way of thinking, as if the first one was worse.

There are many more examples of such type in Polish course books.

II. Description of a diagnostic research

In September 2015 students of three final years of secondary education (17 - 18 year - old) were given a task to solve. They were asked to give as many comments as they could to each step of solving the task. There were 43 students taking part in the research. The students were solving the task under the supervision of a person carrying out the research. They were given maximum of 15 minutes to solve the task. The subject of the task was written down on the board and the students were writing their solutions on the answer sheets. The research task was as follows:

The research task:

Prove that if $a > 0$, *then* $\frac{a^2+1}{a+1} \geq \frac{a+1}{2}$.

The correct solutions of the task (conducted in accordance with the deductive rules) together with the comments, could be following:

$(a-1)^2 \geq 0$	square of any number is not negative,
$a^2 - 2a + 1 \geq 0$	we develop expression using short multiplication formulas,
$2a^2 - a^2 - 2a + 2 - 1 \geq 0$	we convert received expression,
$2a^2 + 2 \geq a^2 + 2a + 1$	we add both sides of an expression $a^2 + 2a + 1$, so that one could once again make use of the short multiplication formulas,
$2(a^2+1) \geq (a+1)^2 / : (a+1)$	we divide both sides by $(a+1)$, not changing the sign of inequality, because of the assumption $a > 0$, so the expression $a + 1 > 0$,
$\frac{2(a^2+1)}{a+1} \geq a + 1 / : 2$	we divide both sides of inequalities by 2,
$\frac{a^2+1}{a+1} \geq \frac{a+1}{2}$	we get an inequality which occurs in a conclusion of the theorem,

so theorem is true.

The research method which was used in the following research was the document analysis of works written by the students.

III. Research results

As a result of using the atomic method to diagnose what actions of a cognitive manner are undertaken by the students while solving the task we came to the description of these actions in a form of a table (which may be called ”the atomic table”). The first column of the table contains students’ single actions (”atoms”). There are 24 of such atoms. Top, first line of the table contains codified symbols of a particular student taking part in the research (U1 - U43). The symbol ”+1” at the intersection of a line and column shows that the activity was noticed and done in a correct way by the student. Symbol ”+0” denotes that the action was not done in the correct way. For example, if a student U13 determined domain of an algebraic expression in a correct way then at the intersection of the third line and 13^{th} column ”+1” was marked in the table (look Appendix).
The bottom part of the atomic table is connected with the global assessment of solutions paying special attention to their correctness. Symbol ”+” which is out of one of the lines means that a student solved the task correctly but did not finish it, solved that task correctly or made a mistake while solving it.

The analysis of the atomic table with the use of the comparative analysis allowed us to synthesize the most or the least often executed activities by the students. Thanks to the atomic analysis it is possible to differentiate two characteristic ways of converting conclusion of a theorem of the research activity. These two ways differentiate the students among one another. One way of solving the activity was connected with both sided multiplication of inequality by expression $(a + 1)$ and by both sided multiplication by 2 so as to reduce the denominator of the inequality. The other way of solving the activity was connected with reducing an expression to a common denominator, converting nominator and denominator on the left of inequality and students were passing from measurable inequality to polynomial inequality.
Both ways of solving the activity led to a correct result. It may be stated that in our case 5 students solved the task correctly using the first method and the second way of correct solving the task was used by 3 students. One can read from the table that there are 8 students who solved the task correctly.
Among the students who took part in the research there was a group of six students who did the task partly correctly and a group of 13 students who made a mistake but were able to finish the task.

Analyzing the atomic table it is obvious that solving the research task demanded 8 - 11 single actions to be undertaken (”atoms”). So the task was not too complicated and the given time was enough to deal with it. That is why it is difficult to say why so few students (8 in 43) solved the task correctly.
There were only 15 students who started solving the research task with quoting the assumptions or verifying them. It might be assumed that only those students were willing to prove the evidence according to the deductive way of thinking. However, none of those students did it. It means that none of 43 students (according to our expectations) started to solve the research task in accordance with the deductive rules.
Another thing which greatly differentiated the students among one another was the action which in the atomic table was called ”determination of the domain of the expression”. It was about excluding the value $a = -1$, while one should remember the assumption $a > 0$ which excluded such value in a natural way. Analysis of students works proves that only ten of them were aware of it.
Students had to perform various actions which were characteristic for algebra and actions connected with letter symbols during solving the research task. These are the actions which students used solving the task:
$a)$ making use of short multiplication formulas,
$b)$ multiplication of inequality by algebraic expression (reflection concerning the change of signs),
$c)$ adding and subtracting expressions to and from both sides of inequality,
$d)$ reducing tangible expressions to a common denominator,

e) reduction of similar expressions,
f) calculating numerical value of algebraic expressions.
The above actions were of different correctness in students works. It turned out that at least 6 of the students used the square of the sum formula for two algebraic expressions in an inappropriate way (look the atomic table in the Appendix). As far as the b) subsection is concerned it may be stated that less than a half of the students who performed the described action did it in a correct way. Converting tangible expressions in a form of inequalities poses a great difficulty to the students. Actions described in subsection c) were performed 34 times and in 30 cases they were performed correctly. As a result, one may claim that adding and subtracting of algebraic expressions is something that students mastered quite well. Reducing tangible expressions to a common denominator (subsection d)) was a difficult any for 2 in 15 students.
The action which is described in the atomic table as "reduction of similar expressions" is a basic algebraic skill, the skill that is used and very often practiced during math lessons. Despite this fact, students still make mistakes while performing these actions. In our case 5 in 31 students made such mistakes.
Four students calculated the numerical value of an algebraic expression (subsection f)) in a correct way.

Summarizing the results of the research, paying special attention to a reductive way of thinking in order to prove some mathematical theorem, it is possible to state that all of the students made use of this way of thinking. However, only one student proved in writing that the way of thinking which was applied by her/him qualified for recognizing this way as a proof argumentation (student U4). We mean realizing the equivalence which has already been mentioned on p. 2 of this work.

IV. Conclusions

The described research, even if it is fragmentary, suggests a few conclusions.

1.The reductive argumentation is a very natural way of proving theorems ([8]) whose thesis is in a form of on inequality or an algebraic expression. Such form of the thesis suggests a student performing algebraic expressions which the student is familiar with, for example, using short multiplication formulas, carrying expressions from one side to the other one, etc. It might be interpreted that this form of thesis of a proven theorem gives a student some tools which may be used in a practical way, thanks to which the students is occupied with some work. While teaching math in a practical way and giving students tasks in which they are supposed to "prove" something and allowing to use reduction, we may create situations in which a student makes an attempt to prove theorems actively and independently.

2. The research analysis shows how troublesome some of the aspects of operations on algebraic expressions are. The most difficult thing is to catch all the assumptions and conditions connected with the term "the field of algebraic expressions" and taking into consideration these conditions throughout the whole process of solving the task. It is worth reminding that only 10 in 43 students carried out this operation correctly. Another activity which posed a difficulty to the students was connected with "converting the expressions on the right side of the inequality", "converting the expressions on the left side of the inequality" and "converting the nominator on the left side of the inequality". These observations suggest that students have not mastered converting expressions satisfactorily.

3. The described research confirms the phenomenon of "automatism of some algebraic procedures" which is described in didactic literature ([10], [2]). This automatism is visible in "compulsion" which students feel in situations which they associate with a known algebraic procedure. In case of the described research, the association of some algebraic expressions with the short multiplication formulas made the students use (correctly or not) those formulas automatically. The mentioned automatism was present in a negative way in the research when the students were performing actions which led them to reduce the inequality to the form of "the unknown quantity on the one side and the known quantity on the other side". This action did not lead to any reasonable result. It could be interpreted in a way that would allow us to notice that the students were going to carry out the procedure which they had known "solve the linear inequality" without paying much attention to the fact whether it was doable and logical.

4. The reductive argumentation, in the described research, turned out to be possible to be used even by an average student. However, it may be stated that the students probably are not fully aware of the conditions which have to be fulfilled so that the reductive argumentation could be a correct argumentation of the probative type. As a result, it is a sign for the teachers that this issue needs more attention during math lessons. Students should have an opportunity to experiment with different, correct and incorrect argumentations. They ought to be able to examine the correctness of such argumentations, supplement the gaps, examine "sufficiency" and "necessity" of conditions in individual inferences. It is very important because the tasks which involve proving have been introduced in the examination papers of the upper secondary school examination at the basic and extended level since 2009. At the extended level of the exam these tasks are connected with different branches of mathematics based on the national curriculum such as: probability theory, sequences, geometry, functions, algebra. The tasks connected with proving were not introduced in the examination papers until 2008. Tasks connected with proving were introduced in the examination papers in 2009 (until the exam in 2015), at the basic level of the exam there are always two tasks of that type, one is connected with geometry and the other one with algebra.
Exemplary tasks from the examination papers are presented

below.

Upper secondary school examination paper (May 2011, extended level):

Prove that if $a \neq b$, $a \neq c$, $b \neq c$ and $a + b = 2c$, then $\frac{a}{a-c} + \frac{b}{b-c} = 2$.

Upper secondary school examination paper (May 2012, basic level):

Prove that if real numbers a, b, c satisfy the inequality $0 < a < b < c$, then $\frac{a+b+c}{3} > \frac{a+b}{2}$.

Upper secondary school examination paper (May 2012, extended level):

Prove that if $a + b > 0$, then the inequality $a^3 + b^3 \geq a^2b + ab^2$ is real.

Upper secondary school examination paper (May 2015, basic level):

Prove that for each real number x and for each real number y the inequality $4x^2 - 8xy + 5y^2 \geq 0$ is real.

The research task which was used in the described research was also taken from the upper secondary school examination paper of May 2010.

5. The didactic literature highlights the tendency of the students to use certain examples to justify the correctness of the mathematical tasks of a general nature. Such situation was also present in this research. A positive observation is the one which shows that only two students (U42 and U43) tried to use that way of theorem verification. The other students were trying to prove the reality of the theorem in a general way.

6. The research which is described in the article made use of the atomic analysis as one of the new research methods of mathematical didactics. This method turned out to be really useful when describing particular ways which students use when solving a given mathematical task. The atomic analysis (with the use of the atomic table) gives a lot of possibilities for various types of comparisons and conclusions.

V. Summary

The conclusions of the research described in this paper suggest the importance of the didactic issue. Tasks connected with proving (according to a new curriculum) should be solved even by average students who graduate from the high school. The tasks from the examination papers prove that. The choice of way of proving (deduction or reduction) should depend on a student. Students should know and be able to use both ways of inference so that they could make use of them in a conscious way. Teachers - practices are responsible for it. Theorists - researchers in the field of mathematical didactics are responsible for showing possible problems that the students may have and possible ways of solving these problems.

I hope that the research described in the article may be the beginning of a greater research which would be the basis for the diagnosis of the students' skills and preferences connected with the use of the reductive argumentation when proving algebraic theorem.

References

[1] J. Bocheński, *Współczesne metody myślenia*, Wydawnictwo W drodze, 1988.

[2] A. Demby, *Typy procedur algebraicznych, stosowanych przez uczniów w wieku 13-15 lat*, Roczniki Polskiego Towarzystwa Matematycznego, Seria V, Dydaktyka Matematyki, 22, (2000), 45–74.

[3] I. Gucewicz-Sawicka, *Podstawowe zagadnienia dydaktyki matematyki*, PWN, Warszawa 1982.

[4] I. Kawecki, *Etnografia i szkoła*, Oficyna Wydawnicza Impuls, Kraków, 1996.

[5] M. Łobocki, *Metody i techniki badań pedagogicznych*, Oficyna Wydawnicza Impuls, Kraków, 2006.

[6] S. Palka, *Ilościowe i jakościowe badania pedagogiczne*, Ruch Pedagogiczny, 1, (1989), 3–3.

[7] S. Palka, *Epistemologiczne podstawy ilościowego i jakościowego badania efektów kształcenia szkolnego*, In: K. Duraj-Nowakowa, J. Gnitecki , (Eds.) *Epistemologiczne wyzwania współczesnej pedagogiki*, Wydawnictwo Naukowe WSiP, Kraków, 1997.

[8] H. Pieprzyk, *Gra jako pomoc dydaktyczna w kształceniu rozumowania redukcyjnego u uczniów klasy IV*, Roczniki Polskiego Towarzystwa Matematycznego, Seria V, Dydaktyka Matematyki, 5, (1985), 7–59.

[9] N. Stehlikova, *Metody badawcze stosowane przez uczestników praskiego seminarium z dydaktyki matematyki*, Roczniki Polskiego Towarzystwa Matematycznego, Seria V, Dydaktyka Matematyki, 21, (1999), 5-13.

[10] S. Turnau, *Wykłady o nauczaniu matematyki*, PWN, Warszawa, 1990.

[11] A. K. Żeromska, *O metodach badawczych stosowanych w dydaktyce matematyki*, Studia Matematyczne Akademii Świetokrzyskiej, 10, (2003), 21–36.

APPENDIX
THE ATOMIC TABLE

	U1	U2	U3	U4	U5	U6	U7	U8	U9	U10	U11	U12	U13	U14	U15	U16	U17	U18	U19	U20
Rewriting the subject of the task	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1
Analysis, development, or using the assumptions	+1	+1		+1			+1	+1	+1					+1					+1	+1
Determination of the domain of the expression		+1									+1		+1							
Treating the expression $(a^2 + 1)$ as a short multiplication formula			+0																	
Both sided multiplication of an inequality by (a+1)	+1	+1		+1	+1		+1				+1	+1	+1	+1						
Conversion of an expression on the right side of an inequality		+1		+1	+1		+1				+1	+1	+1	+1						
Both sided multiplication by 2	+1	+1		+1	+1		+1				+1	+1	+1	+0						
"Carrying" all the expressions on one side	+1	+1		+1	+1	+1	+1	+1	+1	+1				+1	+1	+1	+1	+1	+1	+1
"Carrying" a part of the expressions on one side											+1									
Conversion of an inequality by both sided algebraic conversion												+1								
Reducing to a common denominator						+1		+1	+1	+1						+1	+1	+1	+1	+1
Conversion of expression on the left side of the inequality			+1								+1			+1	+0					
Conversion of a nominator on the left side of the inequality						+1		+1	+1	+1						+0	+1	+0	+1	+0
Conversion of the denominator on the left side of the inequality																			+1	
Reduction of similar expressions	+1	+1		+1	+1	+1	+1	+1	+1	+1	+1			+1		+1	+0	+1	+1	+1
Transition from measurable inequality to polynomial inequality						+1			+1	+1					+0	+1		+1	+1	+1
Noticing the short multiplication formula $(a - 1)^2$	+1	+1		+1	+1	+1		+1		+1										
Counting delta for the left side of the inequality							+1		+1							+1		+1	+1	+1
Counting a zero							+1		+1					+1					+0	
Sketching parabola							+1							+1				+1	+0	
Substitution of the listed roots to inequality																				
Making a sketch different than a parabola									+1											+1
Checking for the changing magnitude "a" if the inequality is real																				
Comment that an expression is not negative.	+1	+1		+1	+1	+1	+1	+1		+1				+0					+1	+1
PARTLY CORRECT SOLUTION									+		+	+	+							
CORRECT SOLUTION	+	+		+	+	+	+	+		+										
SOLUTION WITH AN ERROR			+											+				+	+	+
How many actions were done correctly	8	10	2	9	8	8	11	8	10	8	8	5	5	9	2	6	4	7	10	9
How many actions "atoms" were done	8	10	3	9	8	8	11	8	10	8	8	5	5	11	4	7	5	8	12	10

	U21	U22	U23	U24	U25	U26	U27	U28	U29	U30	U31	U32	U33	U34	U35	U36	U37	U38	U39	U40	U41	U42	U43	Good	Finished
Rewriting the subject of the task	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+0	+1	+1	+1	+1	42	43
Analysis, development, or using the assumptions		+1										+1			+1		+1	+1					+1	15	15
Determination of the domain of the expression		+1	+1							+1	+1								+1	+1	+1			10	10
Treating the expression $(a^2 + 1)$ as a short multiplication formula					+0					+0		+0											+0	0	5
Both sided multiplication of an inequality by (a+1)						+1	+1		+1		+1	+1	+1		+1	+1		+1		+1	+1		+1	21	21
Conversion of an expression on the right side of an inequality						+1	+0	+0	+0	+1	+1		+0		+1	+1		+1		+0	+1			15	20
Both sided multiplication by 2						+1	+1		+1	+1	+1		+1	+1	+1	+1	+0	+1		+1	+1		+1	21	23
"Carrying" all the expressions on one side	+1	+1	+1	+1	+0	+1			+1		+1				+1	+1	+1	+1	+1	+1	+1			30	31
"Carrying" a part of the expressions on one side															+1					+1				3	3
Conversion of an inequality by both sided algebraic conversion							+1	+1																3	3
Reducing to a common denominator	+1	+1	+0	+1	+0														+1					13	15
Conversion of expression on the left side of the inequality							+1	+1			+0		+1	+0	+1	+1	+0				+1			9	13
Conversion of a nominator on the left side of the inequality	+0	+1	+1	+0															+1					9	14
Conversion of the denominator on the left side of the inequality	+1	+1	+1	+1															+1					6	6
Reduction of similar expressions	+1	+1	+1	+0		+0				+1	+1			+0	+1	+1	+0	+1	+1	+1	+1			26	31
Transition from measurable inequality to polynomial inequality			+1																+1					9	10
Noticing the short multiplication formula $(a - 1)^2$																			+0					7	8
Counting delta for the left side of the inequality		+0		+1		+1									+1	+1		+1		+1	+1			13	14
Counting a zero			+0	+1		+1					+1					+0				+1	+1			8	11
Sketching parabola											+1									+1				5	6
Substitution of the listed roots to inequality						+1										+1								2	2
Making a sketch different than a parabola			+0																					2	3
Checking for the changing magnitude "a" if the inequality is real																						+1	+1	2	2
Comment that an expression is not negative.	+1							+1				+1							+0		+0			13	16
PARTLY CORRECT SOLUTION															+			+						6	
CORRECT SOLUTION																								8	
SOLUTION WITH AN ERROR	+	+	+				+	+				+								+	+			13	
How many actions were done correctly	6	8	7	6	1	8	5	4	4	5	9	4	4	2	10	9	3	8	7	10	10	2	5		
How many actions "atoms" were done	7	9	10	8	4	9	6	5	5	6	10	5	5	4	10	10	6	8	10	11	11	2	6		

Quantitative method DEA for economic analysis of efficiency

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Abstract—**This paper discusses Data Envelopment Analysis (DEA) method as a tool for economic analysis in the efficiency evaluation context. The method is adapted for implementation in the education. The reasons, needs, substantiation and suitability for educational application are explained. The paper deals with fundamentals of DEA completed with using, assumptions and interpretation. The broad-spectrum utilization of approach is specified in detail. Furthermore, the most frequently used DEA models are defined. Their principals are described as well as differences. In addition, the attention is paid to the other, more concrete modifications of these models. The conclusion provides interesting findings resulting from DEA employment in the field of education.**

Index Terms—**Data Envelopment Analysis, Efficiency, Decision Making Units, CCR Model, BCC Model**

I. Introduction

On a transnational degree [1], governments put pressure on universities funding. The interest is focused on increasing efficiency of universities' operating activities. For this purpose [16], it is needed to meet and score the cost structure of the educational institutions. The significance of costs, productivity and efficiency evaluation is growing. The total efficiency calculation covers, in fact, the information about operating costs and productivity level.

The availability of efficiency assessment [11] has been reviewed by numerous studies in connection with higher education. The application of various methods indicates that Data Envelopment Analysis is definitely suitable tool for efficiency evaluation of the universities performance. There exist some other techniques [9] for quantification of the efficiency. However, the most convenient one is DEA in the educational field.

Taking into account the high importance of efficiency level appreciation in the education sector context [19], the more progressive mathematical and econometric programming frontier methods have begun to be used not so long ago. The efficiency analysis might be aimed at specific primary school, secondary school, university department, educational program or university in its entirety. The matter of the analysis is in general efficiency specification, its calculation and positive transformation.

It is indisputable [4] that ranking models using multiple characteristics for comprehensive entities illustration are considered as one of the assessment priorities. The rankings describing institutions in education are of great interest. DEA technique enables the preparation of such rankings.

II. DEA approach

The expression Data Envelopment Analysis [6] was first time presented by Charnes, Cooper and Rhodes in the study A Data Envelopment Analysis Approach to Evaluation of the Program Follow Through Experiment in U.S. Public School Education in 1978. This approach was the result of ambition to assess findings of the project Program Follow Through realized under the U.S. Department of Education auspices. Mentioned project represented an extensive effort to use the attempt's statistical axioms and utilize them to a set of comparable schools with the national impact. The research aim was to evaluate supporting educational programs for students placed at a disadvantage which attended some of the public schools in U.S. Charnes, Cooper and Rhodes thus achieved the foundation of the fundamental efficiency measuring method - Data Envelopment Analysis. However, it should be noted that their work was based on Farrell model from 1957. Since DEA was introduced, many modifications and refinements have arisen.

DEA is the technique built on mathematical programming, specifically with the use of linear programming. The beginnings [8] are related to the Operations Research and the Management Science disciplines. Both fields are considerably connected with linear programming. Farrell's concept was extended and the Operations Research evolved fancied quantitative method DEA. The fact is that practice of linear programming becomes respected computing approach for efficiency analysis realization under the various economic areas linked to the decision-making issues.

The purpose of choosing this method [14] is to estimate the efficiency for the relevant data file consisting of so-called Decision Making Units (DMUs). Stated units reach a definite amount of outcomes produced from a given amount of inputs by diverse technological and production operations. The frontier constituting the finest efficiency practice is constructed through DEA. It consists of DMUs characteristic with optimal efficiency within examined set of data. The comparative quantification in the efficiency context is available in this way. The typical attribute for the DMUs belonging to the efficiency frontier is showing the most proper balance between

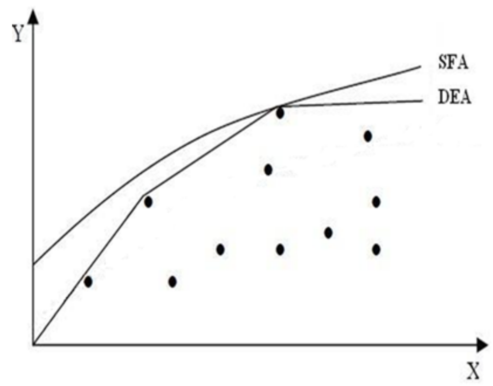

Fig. 1. Efficiency frontier identifying (Source: own adaptation [9])

used inputs and attained outputs. In other words, listed units transform minimal quantity of existing sources to maximal outcome. These DMUs are efficient and have the best balance referred to above. Their best efficiency was reached among all dataset's DMUs. Vacuity between frontier and some DMU indicates the opportunity for improvement to accomplish the main objective in this case - optimal efficiency.

DMUs included in the investigated file data [5] have to be part of homogeneous series. The usual DMUs' operational status corresponds to the previous claim. Nevertheless, the individual ensembles of DMUs from the set of data are tested for homogeneity in many cases. Testing is realized over groups of DMUs, not over single DMUs.

Figure 1 depicts the efficiency frontier for Data Envelopment Analysis stated and explained above. For comparison, the equivalent frontier is represented for another method applied for the purpose of efficiency assessment and also very frequently implemented - Stochastic Frontier Approach (SFA).

III. Broad-spectrum utilization of DEA

Nowadays, Data Envelopment Analysis [7] can be described as an influential analytical and quantitative instrument for assessing the performance. It has registered multi-segmental implementations of many entity types. The principal activities of these entities varied as well as nature of analyzed data.

Frontier efficiency tool DEA [10] is often applied, inter alia, in service industries. Major categories using this analysis for performance improvement are the following:

- education,
- banking, insurance, financial services,
- information technology, media services,
- electricity, water services,
- transportation,
- hotel, restaurants, retail business,
- healthcare, hospital.

The relevant results [13] are achieved for DMUs from non-profit organizations, but also from profit ones. From the DEA approach implementation point of view, the entities factually without revenues are governmental organizations as public schools, libraries or hospitals. Particular cases should be considered, too. Profit production units are quite standard. More specific are entities defined by non-financial revenues or determined by performance where financial results are incorporated just in a small extent. Efficiency assessment is viable by using Data Envelopment Analysis in these instances.

Next advantages of method are potential utilization in the public sector just like in the private sector and data disunity in terms of financial and non-financial inputs and outputs composition. The applications of DEA vary even regarding the territorial aspect. It is fitted for evaluating the performance on the municipal, regional, national or international level.

Studied units can differ in type of business (partnership, corporation, sole trader, Limited Liability Company, cooperative), business activity (selling, buying, investing, renting) and business outcome character (products or services).

IV. The most frequently used DEA models

The original DEA method [15] has a numerous variations that offer the elimination of the possible weaknesses emergence. The attention is paid to the two DEA models in this contribution - the basic one and its adjustment. Specifically, the two introduced models, ranked to the classic DEA models category, are:

- CCR model,
- BCC model.

A. *CCR model*

CCR model is the basic DEA model named after Charnes, Cooper and Rhodes. They constituted it in the seminal work in 1978. CCR model is one of the most recurring models linked to the Data Envelopment Analysis. The reason is its elementary DEA background through which model provides platform for other more sophisticated versions.

The point is [12] to accomplish the maximal level of all outputs / the minimal level of all inputs taking into consideration the constant returns to scale. There exists an assumption that inputs enlargement of the efficient DMU (DMUs) by some agent should cause attributable outputs increase by the identical agent.

Constant returns to scale reflected in CCR model entail rejection of negative or even positive economies of scale existence. Returns to scale in constant form then signify premise of analogical operational efficiency for small entities and large entities, too.

The resulting efficiency [18] is formed by the ratio of the weighted sum of outputs to the weighted sum of inputs. Valid condition is that the efficiency ratio belonging to any unit from the data set is not more than one. It can take value from zero to one.

The typical feature, for all DEA models categories, is identification of such DMU (DMUs) which signals the best inherent efficiency from the perspective of conversion inputs into outputs. Subsequently, all other DMUs are sorted correlative to this most efficient DMU (group of DMUs) defined previously.

B. BCC model

BCC model [17] represents the approach called in abbreviated form after its founders Banker, Charnes and Cooper. It assumes variable returns to scale. This model is narrowly associated with basic CCR model and states its extended version.

It is feasible to express the efficiency in figures on input orientation basis as well as on output orientation foundation. It makes no difference between constant returns to scale and variable returns to scale, i.e. both orientation types are admissible to apply for model with constant returns to scale (for example CCR model) and also for model presuming variable returns to scale (for example BCC model). Thus, discussing CCR and BCC model, there exist two main variants. Each of them is frequently used in empirical studies. Selection depends on objective which is pursued. Input-oriented models dispose certain pros and cons. The same is putted into effect for output-oriented models.

Input-oriented models [3] are focused on inputs with the fact that they seek to achieve a minimal available resources quantity regarding fixed amount of outputs. On the other hand, output orientation means an effort to reach the maximal outputs. It is important to point out the condition of constant inputs simultaneously fulfilled. A fundamental divergence of these approaches is therefore their prime concern in meeting determined level of outputs or inputs.

In the connection with model orientation [2], disparities occur. They concern the results of CCR and BCC models in changing their orientation. CCR model findings show similarity in the both samples (output-maximized, input-minimized). This does not apply to BCC model variants.

V. Conclusion

It is a matter of fact that higher education sector is characterized by some specific features. These include its non-profit nature, conversion of multiple inputs to multiple outputs or inputs and outputs prices deficiency. It makes challenging to measure efficiency of the higher education institutions.

Many methods employed for this purpose were, unfortunately, inappropriate choice. Considering this fact, it is much more significant that DEA fulfills assumptions for use to the higher education production process. In addition, DEA generates relevant results about efficiency level. Numbers of empirical studies showed that research conclusions concerning efficiency calculations from the education field were quite distorted by using some mathematical-statistical approaches. Also, DEA findings are applicable in practice and provide information about possible improvement for each examined institution of education segment. Progress of the examined educational institution (unit) can be reached over comparison with the other unit, evaluated as an efficient. The comparative unit represents university or even particular department/institute of the university.

Complications in assessing the efficiency tend to rise in the investigation of data set comprising variables given in different units. DEA models can handle mentioned variety. Other benefit sticks in less usage limitation in compare with several methodological platforms targeted at efficiency analysis.

The efficiency measuring by standard performance indicators is the topic of controversy. On the one hand they are very popular and easy to use, but on the other hand these indicators are considerably simplified. Among their cons the necessity of comparison a lot of results is involved. Economic analysis of efficiency applied to institutions of the education could be realized under partial ratios computations. However, assessment would include just a certain entity' activities and it would be accompanied by a fragmentary results. Data Envelopment Analysis is a technique for evaluating the performance of a set of units from the complex perspective.

The upshot formulation is not really complicated summarizing the results obtained by DEA. This frontier analysis type can be characterized by high comprehensibility degree, transparency and relatively simple interpretations. It allows the admittance of competent persons to the helpful information for potential exploitation in the decision making process to increase the efficiency of the operation linked to the educational institutions or the entire sector.

The value of overall efficiency is an important indicator in order to form a view about functioning of educational entities. Despite disadvantages, which DEA has as well as further methods, it is a convenient tool for universities' efficiency measuring.

Acknowledgment

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References

[1] M. Abott, C. Doucouliagos, *The efficiency of Australian universities: A Data Envelopment Analysis*, Economics of Education Review, 22(1), (2003), 89–97.

[2] N. Adler, L. Friedman, Z. Sinuany-Stern, *Review of ranking methods in the Data Envelopment Analysis context*, European Journal of Operational Research, 140(2), (2002), 249-265.

[3] C. Bampatsou, G. Hadjiconstantinou, *The use of the DEA method for simultaneous analysis of the interrelationships among economic growth, environmental pollution and energy consumption*, International Journal of Economic Sciences and Applied Research, 2(2), (2010), 65–86.

[4] M. L. Bougnol, J. H. Dulá, *Validating DEA as a ranking tool: An application of DEA to assess performance in higher education*, Annals of Operations Research, 145(1), (2006), 339–365.

[5] A. S. Camanho, R. G. Dyson, *Data Envelopment Analysis and Malmquist Indices for measuring group performance*, Journal of Productivity Analysis, 26(1), (2006), 35–49.

[6] W. W. Cooper, L. M. Seiford, K. Tone, *Data Envelopment Analysis: A comprehensive text with models, applications, references and DEA-solver software*, Springer, New York, 2007.

[7] W. W. Cooper, L. M. Seiford, J. Zhu, *Handbook on Data Envelopment Analysis*, Springer, New York, 2011.

[8] C. Daraio, L. Simar, *Advanced robust and nonparametric methods in efficiency analysis: Methodology and applications*, Springer, New York, 2007.

[9] E. Drabiková, *Data Envelopment Analysis as a tool for evaluation of the banking efficiency*, GRANT Journal, 4(1), (2015), 19–22.

[10] A. Emrouznejad, E. Cabanda, *Managing service productivity: Using frontier efficiency methodologies and multicriteria decision making for improving service performance*, Springer, New York, 2014.

[11] J. Johnes, *Data Envelopment Analysis and its application to the measurement of efficiency in higher education*, Economics of Education Review, 25(3), (2006), 273–288.

[12] R. Madlener, C. H. Antunes, L. C. Dias, *Assessing the performance of biogas plants with multi-criteria and Data Envelopment Analysis*, European Journal of Operational Research, 197(3), (2009), 1084–1094.

[13] M. Rahimian, M. Soltanifar, *An application of DEA based Malmquist Productivity Index in university performance analysis*, Management Science Letters, 3(1), (2013), 337–344.

[14] J. R. San Cristóbal, *A multi criteria Data Envelopment Analysis model to evaluate the efficiency of the renewable energy technologies*, Renewable Energy, 36(10), (2011), 2742–2746.

[15] D. Schaar, L. Sherry, *Comparison of Data Envelopment Analysis methods used in airport benchmarking*, In: Proceedings of 3rd International Conference on Research in Air Transportation - ICRAT, Fairfax, 2008.

[16] E. Thanassoulis, M. Kortelainen, G. Johnes, J. Johnes, *Costs and efficiency of higher education institutions in England: A DEA analysis*, Journal of the Operational Research Society, 62(7), (2011), 1282–1297.

[17] K. Tone, B. K. Sahoo, *Evaluating cost efficiency and returns to scale in the life insurance corporation of India using Data Envelopment Analysis*, Socio-Economic Planning Sciences, 39(4), (2005), 261–285.

[18] M. Wen, H. Li, *Fuzzy Data Envelopment Analysis (DEA): Model and ranking method*, Journal of Computational and Applied Mathematics, 223(2), (2009), 872–878.

[19] A. C. Worthington, *An empirical survey of frontier efficiency measurement techniques in education*, Education Economics, 9(3), (2001), 245–268.

Evaluation of the educational process from the perspective of quality management

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***Abstract*—The use of modern methods of management enables higher performance. Human resources management is an essential part of any organization or community, whether it is a manufacturing company or institution. Education involves people at every step. Education is a complex process consisting of pre-established requirements of both parties. Thus, the assessment of this process often takes place only by the trainer, and disregard the needs of trainees. Evaluation of the processes of teaching should be carried out from two perspectives. First, an aspect of the trainer, his progress in learning, skills and attitudes. It involves also his capabilities and various other factors, which are related with the trainer and his personality. The second aspect is the situation of trained personnel, their experience, willingness to learn, imagination, motivation to achieve results and other factors The principles of Total Quality Management (TQM) application in education has encountered an attractive and growing importance. The focus should be on evaluating the results of the training program. One powerful tool of TQM, the Quality Function Deployment (QFD) is used in educational programs and has been successfully applied in the field of employee's education. Assessment of the presented educational process is conducted in both aspects. They take into account the objectives of the trainer but also the requirements of staff. The presented paper deals with trainings in different companies where employees are trained by lecturer to work with forklift. Evaluation of the education process is the feedback for the trainer and for its further educational activities and for the process of transferring knowledge and skills. Evaluation of the results of the training is conducted primarily on a quantitative basis. It is necessary to highlight a qualitative basis, where the results can include usability and the need for trained staff in practice and transfer of experience to other employees in practice.**

***Index Terms*—Education, Process Control, Quality Management, Quality Function Deployment**

I. Introduction

Evaluation of the education process represents the feedback for the trainer and for its further educational activities and the process of disseminating knowledge and skills. This forms a space for obtaining information for the next step of education.

Evaluation of training results is realized mainly on a quantitative basis. It is necessary to highlight the qualitative basis, where the results in practice could include usability and the need for trained staff and transfer of experience to other staff.

The use of advanced process control methods allows to achieve higher performance. Management of human resources is an integral part of every organization. Employees, and thus the trained staff, become one of the factors of competitiveness of the organization. Organizations should be flexible in learning, promptly responsive to the changing demands of the market. Therefore, there are multiple approaches to learning, where the most widely used is Kirkpartick's model shown in Fig. 1.

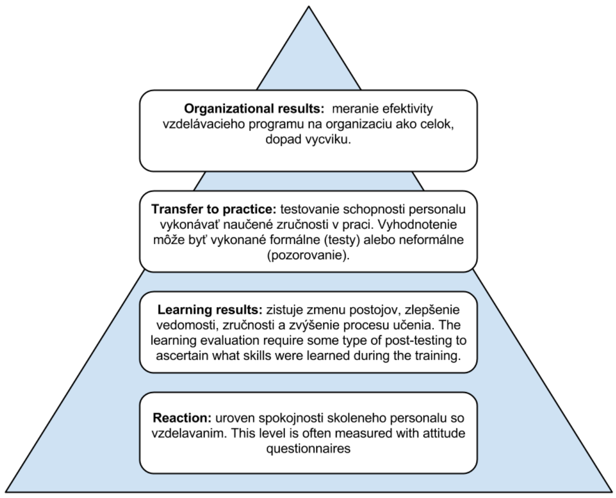

Fig. 1. The levels of learning evaluation according to Kirkpatrick

Kirkpatrick says, "Trainers must begin with desired results and then determine what behavior is needed to accomplish them" [9]. Based on Kirkpatrick's model were developed other models, which are considering the feedback and control (Humblin's 5 levels, Bringkenhoff's 6 levels) [9].

A. The role of people in organizations

Organisations willing to achieve success in the next period should primarily be active learners and promptly responsive to changing market demands. Therefore, it is important to understand that education contributes to the achievement of long term objectives. One of the masters of Japanese quality management Ishikawa argued that quality control begins and ends with upbringing and education. Learning organization is supported by the EFQM Excellence Model, where as learning organization we consider each organization where employees are voluntarily undergoing a process of continual expanding of their knowledge and skills to improve their own work and improve the performance of the organization [1], [3], [14].

Learning organization model is based on four basic characteristics and we can depict it in Fig. 2.

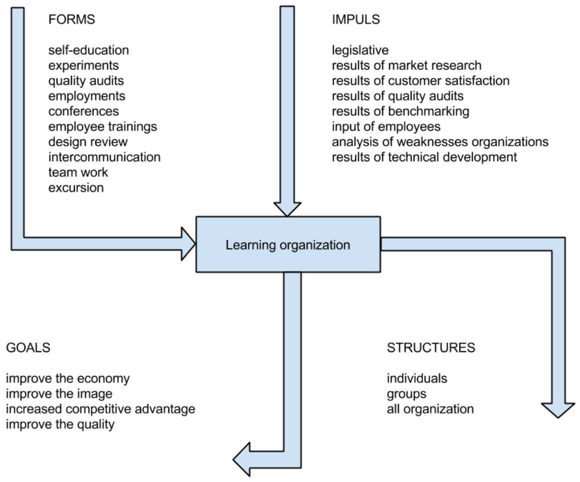

Fig. 2. Schema of learning organization

In the field of corporate education has application of the TQM principles attractive and growing importance. The company management recognized the need to refine the research tools used for training and to increase the quality of education. The focus should be on evaluation of the educational program results. One powerful tool of TQM - Quality Function Deployment (QFD) is used in educational programs and has been successfully applied to the field of emplyees education.

The current study applies QFD to evaluate the education quality of company whose employees are being educated to operate forklift. The study was conducted with three main objectives:

1) examine the differences between quality attributes required to personnel and signs of of quality provided by trainer who conducted the training,
2) understand the most and least important elements of training in terms of future forklift employees,
3) describe quality requirements in relation with important working elements.

II. Methods

In the literature, there are many proposals for quality attribute measurement methods [1], [2], [4], [5], [6], [7], [8], [13], [3], [9], [11], [12], and they differ from one another in attributes.

The ultimate goal of QFD is to translate customer requirements - the so-called "Voice of customer" (VOC) - into quality characteristics of the final product or service [15]. One of the products of QFD is a "House of quality" (HOQ) matrix, which enables a quick visual comparison of "what customer want" versus "how suppliers can give it to them" [5].

The House of Quality correlates desired qualities to a large variety of means by which customer desires can by satisfied. The HOQ matrix arranges important data in such a way that key quality issues, their interrelationships, and their significance relative to one another are readily discernible, it also establishes criteria for successful customer satisfaction.

A. *Quality Function Deployment*

QFD is a planning tool that can be used to translate customer needs and expectations into appropriate design actions. The real value of QFD is ability to direct the application of other quality tools to those design tasks that will have the greatest impact on the team‘s ability to design a service that satisfies the needs and expectations of the customers, both internal and external [17].

The VOC items will be developed into a list of needs used later as an input to a relationship diagram, which is called QFD's House of Quality. It is a series of diagrams, in maximal version it is about 35 of them. The most important and most used is the first one. House of Quality has 8 fields in total. From perspective of companies which do not represent the world top it is enough to realize only the first diagram QFD in simplified way, where 5 fields are used, like in literature [4].

The basic construction of House of Quality is depicted on Fig. 3.

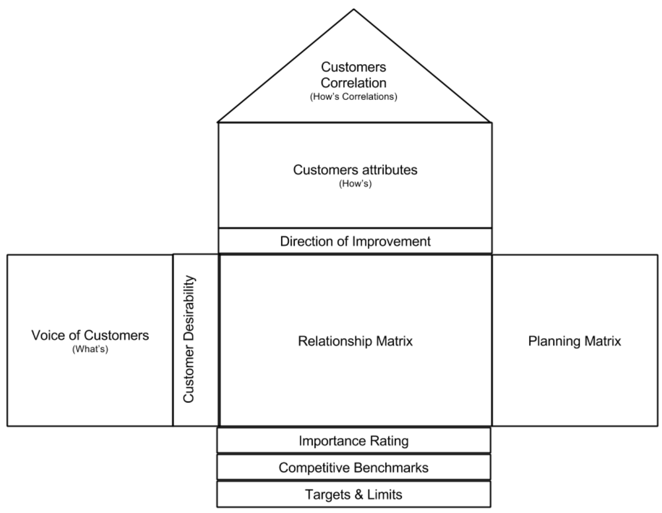

Fig. 3. House of Quality

The field Voice of Customer is subjective, qualitative and non-technical voice of customer ("What's"). These are the results from the survey of second questionnaire. Here is added also the importance of the individual requirements from the customer point of view. In our case these are the results of the second survey ordered by modus.

Field How's represents the translation of customer requirements to service requests. The columns on the top of the center box are attributes of management of restaurant ("Hows"). These are the results from the survey of first questionnaire.

The center box of the House of Quality is a Relationship matrix diagram. The quantitative relationship between "What's" and "Hows" is the key information produced in a QFD study. Below the matrix is computed the Importance of customer Attributes, i.e. importance is multiplied by the modus in every column and values are summed.

In planing matrix is carried out the comparing of competitors from the customer perspective. Customer is comparing the fulfillment of his needs by three providers, but do not know, what providers they are.

In Importance rating is expressed the weight of each service requirement. If two requirements are significant and complex and the roof of the house indicates the interaction, then the most attention is dedicated to them when developing.

Field Competitive benchmarks is made using standard tests. Compared are three services - our service and two competitive services.

Field Targets and Limits performs quantification of requirements for a new product. Based on values from ratings derives requirements for new - better product.

Roof expresses the conflict of interests. Here are examined all couples (pairs) How's with respect to the interactions, whether it is the weakening or amplification of influence. This allows early identification of harmful interactions between the particular specifics, what allows prevent conflicts of interests. Each cell in the roof is a measure of the possible correlation of two different "Hows". This is a very important function in the QFD because "Hows" are most often correlated. The following question for this part of the House of Quality helps to clarify the relationships among requirements: "If one requirement is improved, will it help or hinder another requirement?" In this case there are not a lot negative correlations in the roof, so we can state that the important customer requirements were selected.

III. Measurement

In this article was used QFD method to measure the realized satisfaction with the education process. Assessed was the lector, educational process and information of educational process. The other side of assessment were employees and the dedicated staff of the given training, which was led by the lecturer [10].

In 1. Field, Factors, were questioned 11 factors. The 2. Field is modus for particular factors. The 3. Field, Sectors, was splitted by 4 questions to lecturer, process and informations.By this field is shown orientation towards growth or decline. In this case, were selected such questions, which are expected to only increase quality. The 4. Field is the dependency between the various questions in the field of sectors. The 5. field is a relationship between the first and second fields.

Resulting matrix of QFD method is on Fig 4, where following fields were used: Customer Attributes, Customer Requirements, Relationship matrix, Importance ratings and Customer Correlation.

Rows QDF, Factors, are $X_1, X_2, ..., X_m$, $x_1, x_2, ..., x_m$ are the corresponding scores for $X_1, X_2, ..., X_m$. Values $y_1, y_2, ..., y_n$ are corresponding scores for $Y_1, Y_2, ..., Y_n$. The relationships between y_j and x_i are given in the following equation:

$$y_j = \sum_{i=1}^{m} p_{ij} x_i \qquad (1)$$

In Fig. 4 the column Trainer expertise in given topic goes the scores for modus 3, what is X_i, and the scores inf the column are corresponding x_i. The row "absolute importance" corresponds to the scores of key customers, that is, y_i, and they are calculated based on equation (1), as follows:

$$y_1 = \sum_{i=1}^{12} p_{ij} x_i = 1 \times 9 + 1 \times 1 + 1 \times 9 = 19 \qquad (2)$$

Such method is used for calculation the values of absolute importance for all columns.

A. The result

The result of QFD are critical parameters for the employees. We typically select $20 - 30\%$ of parameters with the highest calculated value (Pareto principle). These parameters we assign the top priority, they are crucial for competitiveness and to improve the quality perceived by employees.

The center box of the House of Quality was completed using the brainstorming method, where the interdependencies between Factors of employees and Sectors were assessed applying the rating scales $(9, 3, 1)$ defined as the Matrix Weight. Brainstorming in group of people was used also when completing the roof of the House of Quality assessing the interdependencies between Sectors. In making this assessment were used trade offs for 5 dependencies.

For the employees is the most important element the Possibility of trying out theories in practice. These are employees, for whom it is important to try out the gained knowledge, so they can react. It is a convenient way to find additional questions and comments on training. This locality belongs under the training process. Trained staff judged this as the most critical area and was least satisfied with it. The reason for the negative evaluation could be physical and mental disposition of trained individuals. Trained staff tend get jitters at the beginning of the practical teaching, which may cause forgetting the gained information during the forklift operation. Subsequently, the trainer tries to recall basic information and renew acquired knowledge of trained personnel. This process of updating knowledge is realized at the expense of the time that has been allocated for the practical part of the course. Another reason is the difference in manual skills, which is also affected by the fact that the trained individual is or is not the owner of a driving license.

The second element is the Sufficiency of information on the theoretical part - test. The area was investigated in terms of trainer. Staff being trained judged this area as the second most critical. The assessment therefore shows that they were not satisfied with the amount of information provided to prepare for the test from theoretical part. The reason for this high

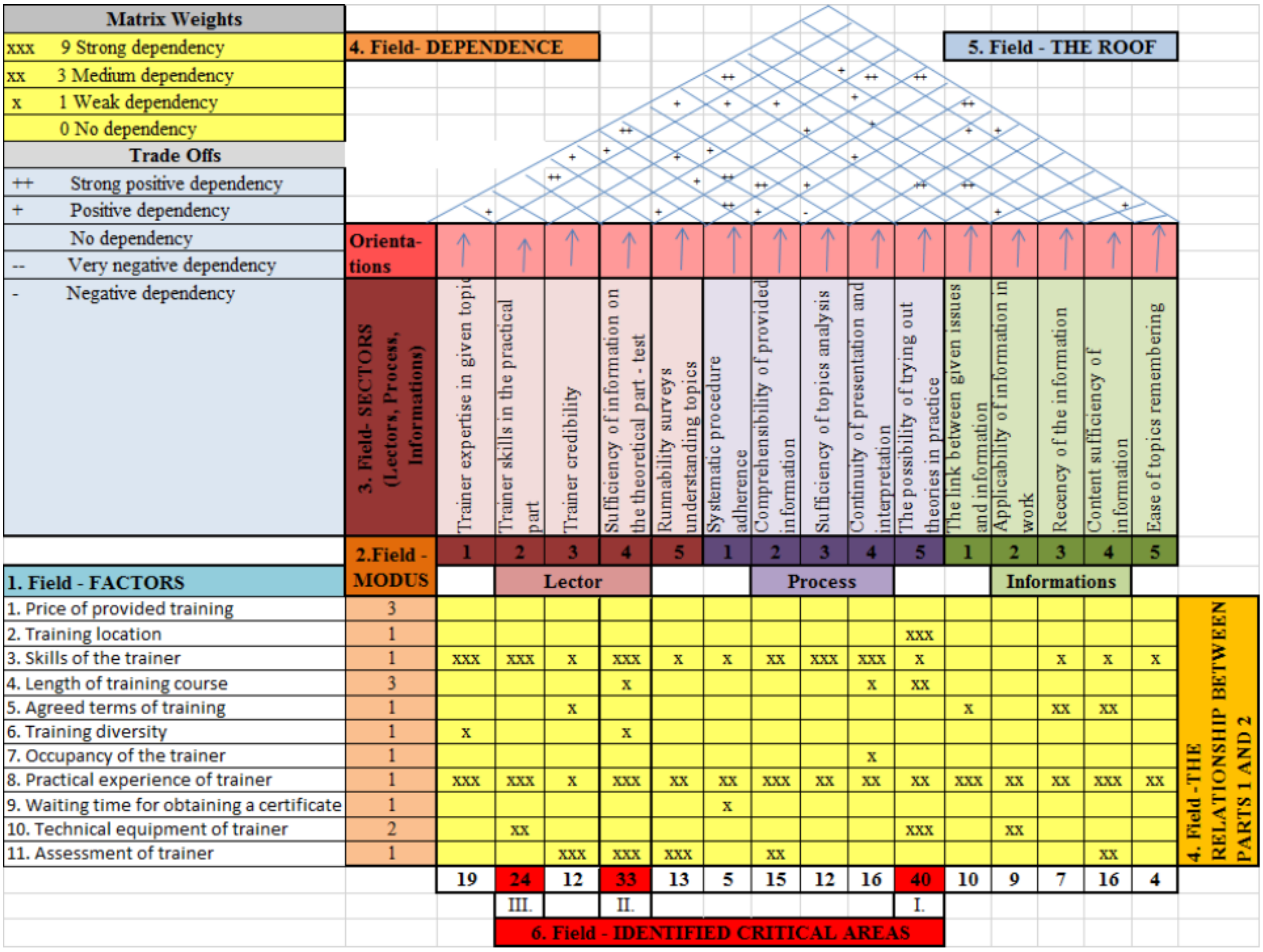

Fig. 4. House of Quality for educational process

rating may be the lectures about regulations and standards that have more complex form and need to be memorized. It is not possible to explain everything in practical examples to easily save the information in memory and so may the staff being trained gain a feeling of insufficient amount of information, even though that information was provided in sufficient quantity, but some in more complex form.

Third major factor is the Trainer skills in the practical part. The area was investigated in terms of the trainer and was rated as the third most critical with respect to the satisfaction of personnel in training. One of the first step in launching the practical part is the description of the elements of a particular forklift, then finding his state functions and practical demo of operating the forklift by trainer, which is usually done once. The trainer then gives space to qualified personnel to adoption and application of acquired knowledge and skills in practice. It is a space mainly designed for the staff being trained and the acquisition of practical manual skills.

IV. Conclusion

This article discusses the assessment methods of survey questionnaires to determine employees satisfaction with educational proces. The results can be applied in various educational processes, because assessed is not only the trainer, but also staff training requirements are taken into account.

In QFD the array of “What’s” represents the customer attributes. This “What’s” are “spoken” by the customer and are called “Performance Quality”. Management should seek to generate more elements with performance quality (Kano’s model of customer satisfaction) [15].

Acknowledgment

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References

[1] M. Benková, Ľ. Floreková, G. Bogdanovská, *Systémy riadenia kvality*, Elfa, Košice, 2007.

[2] L. Bernal, U. Dornberger, A. Suvelza, T. Byrnes, *Quality Function Deployment for services: Handbook*, International SEPT Program, Universitat Leipzig, Leipzig, 2009.

[3] M. Ďurišová, A. Kucharčíková, E. Tokarčíková, *Assessment of higher education teaching outcomes (Quality of higher education)*, Procedia - Social and Behavioral Sciences, 174, (2015), 2497–2502.

[4] P. Blecharz, H. Štverková, *Využití metody QFD při řízení kvality ve službách*, Kvalita, 6(1), (2010), 48–52.

[5] D. P. Clausing, *Total quality development: A step-by-step guide to world-class concurrent engineering*, Don Clausing, ASME Press, 1994.

[6] J. J. Cronin, S. A. Taylor, *Measuring service quality: A re-examination and extension*, Journal of Marketing, 56(3), (1992), 55–68.

[7] E. Cudney, C. C. Elrod, A. Uppalanchi, *Analyzing customer requirements for the American Society of Engineering Management using Quality Function Deployment*, Engineering Management Journal, 24(1), (2012), 47–56.

[8] Ch.-Ch. Chen, *Integration of quality function deployment and process management in the semiconductor industry*, International Journal of Production Research, 47(6), (2009), 1469–1484.

[9] D. Kirkpartick, *Evaluating training programs: The four levels*, 3rd ed., Berrett-Koehler Publishers, San Francisco, 2006.

[10] S. Lištinská, *Monitorovanie kvality vzdelávania*, Bakalárska práca, TUKE, FBERG Košice, manuscript, 2015.

[11] P. Kotler, P., *Marketing management*, 10rd ed., Grada Publishing, Praha, 2001.

[12] P. Kotler, G. Amstrong, *Principles of marketing*, Pearson Education, 2010.

[13] S. Mal-Kong, *Relationship between service gap and perception for mapping of quality attributes into four service quality factors*, International Journal of Services, Economics and Management,3(1), (2011), 36–61.

[14] F. D. Orel, A. Kara, *Supermarket self-checkout service quality, customer satisfaction and loyalty: Empirical evidence from an emerging market*, Journal of Retailing and Consumer Services, 21, (2014), 118–129.

[15] M. Pavlíčková, *Evaluation of customer satisfaction using the Quality Function Deployment*, In: Proceedings of the 16th International Carpatian Control Conference (ICCC), May 27 - 30, 2015, Danvers, Hungary, (2015), 370–373.

[16] K. R. Roy, *A primer on the Taguchi Metod*, 1 ed., SME, Michigan, 1990.

[17] K. Yang, *Voice of the customer: Capture and analysis (Six SIGMA Operational Methods)*, 1 ed., McGraw-Hill Professional, 2007.

A selected part of graph theory in economic and education practice

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***Abstract*—One of the youngest parts of discrete mathematics is a graph theory. Although its history is dated only back to 18th century, it has extensive usage in many fields. Here we want to point out to its utilization in the area of economics and how our students are able to use it in order to solve different tasks they come in contact with in their practice.**

***Index Terms*—Graph Theory, Economics, Decision Trees**

I. Introduction

In mathematical analysis a graph of a real function f of one variable x represents the set of all points $[x; y]$ such that $x \in D_f$ and $y = f(x)$, where D_f is a set of all $x \in \mathbb{R}$ for which $f(x)$ makes sense. Despite of that in the graph theory - part of discrete mathematics, a graph $G(V, E)$ represents a set of points - vertices, and set of curves - edges. Whenever there is an edge between two vertices there is a curve connecting the points representing the coresponding vertices in the drowing of the graph. The vertex set of a graph G is denoted as $V(G)$ and the edge set $E(G)$.

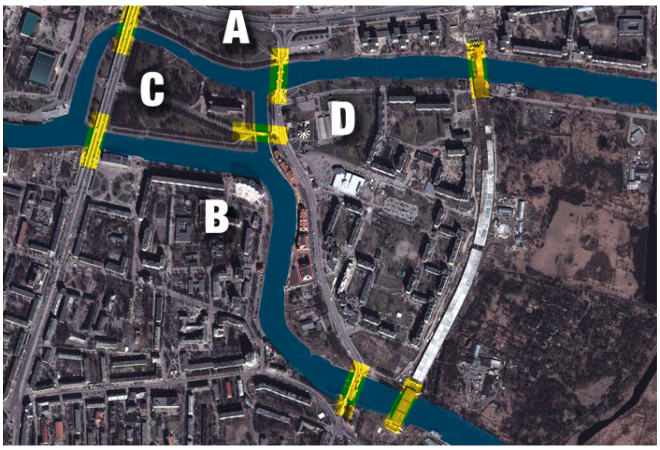

Fig. 1. Seven bridges over Pregel in Königsberg [14]

The birth of graph theory can be dated to the beggining of eighteenth century and it is connected with famous mathematician Leonard Euler. His solution of the Königsberg bridge problem (1736) is considered to be the first theorem of graph theory and the first proof in the theory of networks [15].

There were seven bridges in Prussian city Königsberg (nowadays Kaliningrad in Russia) over the river Pregel. The situation is depicted on Figure 1. The inhabitants wanted to walk through the city, cross each bridge only once, so that the two river islands could only be reached by the bridges and every bridge that was once accessed, had to be crossed to its other end. The starting and ending points of the walk needed not to be necessary the same.
All the practical experiments to solve this task came into fault. Euler pointed out to the fact that the walk inside each land-part is irrelevant. The important is only the sequence of the crossed bridges. Hence, he reformulated this topological problem using a lot of abstraction - he replaced each land-piece by a vertex and each bridge by an edge and by getting the resulting mathematical structure called a graph, he grounded the base of graph theory. With a newly developed technique of analysis Euler proved that the Königsberg bridge problem has no solution.

Since then graphs found widespread use as models, controlling and decision tools in many areas.

Graphs are good representations for roads, underground or railways nets - see Figure 2. In these models the cities (villages, underground stations, bus stops,...) are represented by vertices and the roads, underground lines or railways by edges connecting them. Sometimes, it is important to consider an orientation of the connections (one-way roads), hence, in the graph distinghuish between the edge $e = uv$ and the edge $e' = vu$; or consider multiple connections between 2 points of interest. In such a case we define the orientation for each edge of a graph G and we are speaking about oriented graph; or multigraph in the second case (multiple edges are allowed). Thanks to that graphs have extensive use in logistics.

Graphs have usage in informatics as well - they provide good models for network topology. The parts of electrical circuits can be represented via graphs too.

Surprisingly, graphs found their applications not only in mathematics, informatics and related natural sciences, they can be used in social sciences as well. In social studies, by some abstraction, people can be replaced by vertices

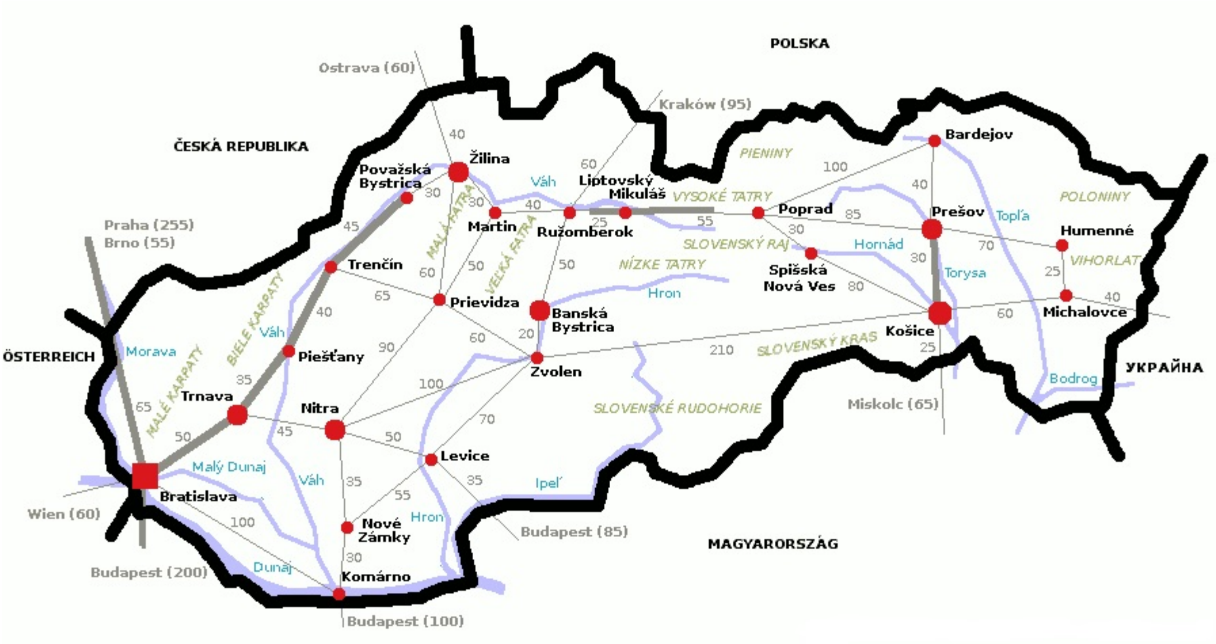

Fig. 2. The main traffic nodes of Slovakia [20]

and relationships between them as edges. Hence, also social science problems can be solved via graph theory tools.

Here we point out to usage of graph theory in economic and education practice.

II. Implementation of the Graph Theory in Economics

The graph theory, with its manifold tools, is incredibly helpful discipline considering focus on the economic area. Many tasks from economic practice can be solved by using some instruments for decision making. These are combined with utilization of other disciplines utilities. Some of the collaborative branches are mathematics, statistics or econometrics.

Adoption of the correct decisions is strategically highly important step and effective decision making is one of the crucial challenges arising from the complexity, diversity and multidisciplinarity of the economic questions at the present. It should be noted that another important factor affecting the various economic decisions is uncertain character of the national economics background. The decision trees (see example on Figure 3), amongst others, are applied to resolve the issues.

Decision trees [2] are the most favorite predictive techniques. The point is the representation of the connections among particular variables. These models are helpful through meeting classification difficulties as well as prediction ones. Generally, the cases are divided into two standard groups - positive or negative. When we consider the churn classification, instances are matched with group "churn" or "non-churn". Decision tree models are processed from the general to the particular matters in the context of the representation and

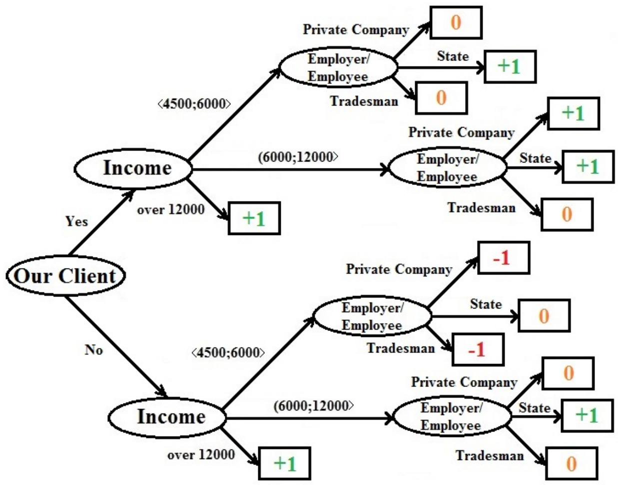

Fig. 3. Credit scoring decision tree (Source: Authors research in [4])

assessment, therefore, they are often represented by oriented graphs.

Great use and potential of decision trees is registered in many fields of economics such as project management, risk management, retail business, corporate finance, marketing, capital investments, capital budgeting or banking and other financial services. Decision tree models provide support for organizations in decision making for the most valuable client retention [11], [16], more accurate targeting and acquiring new clients [9], computerized credit scoring and approval [4], [7], fraud detection and prevention [4], [5], [18], offering segment focused products or services [12], diverse analysis aimed to the clients [6], churn detection and prevention [4], [8], [10], more effective marketing [13], better relationship [17], transaction patterns [19] and last but not least suitable risk management [1], [3].

Figure 4 depicts some of the economic issues taking into consideration financial services sphere. Concerning stated instances, decision trees are highly appropriate tool for facilitating decision making process.

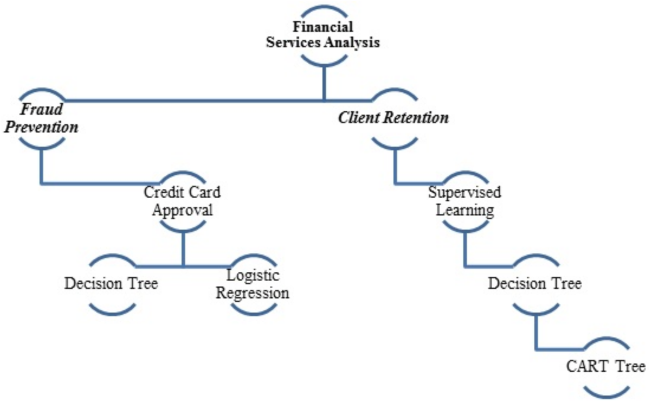

Fig. 4. Selected economic tasks solvable under decision tree models utilization (Source: Authors adaptation)

III. Graph Theory at Technical University of Košice

As a lot of problems of real life situations can be transformed into the speach of graph theory and solved via its methods, the basics of graph theory have their accepted place in the engineering education. At the Technical University of Košice, Košice, Slovakia, depending on the faculty, students came in touch with the graph theory as a separate subject (e. g. at the Faculty of Economics), or in an incorporated form into another compulsory subject (e. g. the Theory of Decision Making at the Faculty of Mining, Ecology, Process Control and Geotechnology) or elective subject (e. g. Applied Mathematics at the Faculty of Mechanical Engineering).

Even if the curriculum of education at the Technical University of Košice contains only basics of graph theory, our students are able to use the obtained knowledge in their study and research.

The tools of graph theory were used in order to identify critical points in production processes and with the aim to optimize the production and transportation processes of a given company, for example to identify the faults causing damages on conveyor belts during the transportation of coal.

In several works the main aim was to find an optimal strategy or way, respectively a way with given properties, for example in a study where the graph theory helped to determine an optimal emergency-exit-way in a case of fire from every store and office of a given building.

Very often graphs were used by our students with connection to economy for modelling of economic processes and market situations (e. g. the state of the real estate market in Košice), for business controlling, decision making based on probabilistic decision trees and elsewhere.

IV. Conclusion

In the present paper we showed an utilization of a selected part within graph theory, a significant element of decision theory. The utilization was connected with economic and education practice. In the two main sections we have described applications in economics and in students practice.

Widespread use of decision tools was justified by numerous different fields and instances from economic practice. Emphasis was placed primarily on decision trees.

We have shown that the implementation of graph theory into engineering curriculum is highly relevant as students are able to use it in order to solve many different problems. These were demonstrated by some examples of solved tasks by students of the Technical University of Košice that reflect the actual situation in practice.

Even if the graph theory has widespread use, our students use only a fraction of it mostly because of missing knowledge caused by not adequate time donation for graph theory during their university study. We believe, that its increase would have positive effect on graph theoretical and practical solutions reached at student's side.

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References

[1] A. Ahmed, B. Kayis, S. Amornsawadwatana, *A review of techniques for risk management in projects*, Benchmarking: An International Journal, 14(1), (2007), 22–36.

[2] K. Chitra, B. Subashini, *Data mining techniques and its applications in banking sector*, International Journal of Emerging Technology and Advanced Engineering, 3(8), (2013), 219–226.

[3] P. K. Dey, *Project risk management: a combined analytic hierarchy process and decision tree approach*, Cost Engineering, 44(3), (2002), 13–27.

[4] E. Drabiková, E. Fecková Škrabuľáková, *Decision trees - a powerful tool in mathematical and economic modeling*, manuscript, 2016.

[5] P. A. Estvez, C. M. Held, C. A. Perez, *Subscription fraud prevention in telecommunications using fuzzy rules and neural networks*, Expert Systems with Applications, 31(2), (2006), 337–344.

[6] X. Hu, *A data mining approach for retailing bank customer attrition analysis*, Applied Intelligence, 22(1), (2005), 47–60.

[7] H. Ince, B. Aktan, *A comparison of data mining techniques for credit scoring in banking: A managerial perspective*, Journal of Business Economics and Management, 3, (2009), 233–240.

[8] M. Kaur, K. Singh, N. Sharma, *Data mining as a tool to predict the churn behaviour among Indian bank customers*, International Journal on Recent and Innovation Trends in Computing and Communication, 1(9), (2013), 720–725.

[9] J. W. Kim, B. H. Lee, M. J. Shaw, H. L. Chang, M. Nelson, *Application of decision-tree induction techniques to personalized advertisements on internet storefronts*, International Journal of Electronic Commerce, 5(3), (2001), 45–62.

[10] D. A. Kumar, V. Ravi, *Predicting credit card customer churn in banks using data mining*, International Journal of Data Analysis Techniques and Strategies, 1(1), (2008), 4–28.

[11] B. Lariviere, D. Van den Poe, *Predicting customer retention and profitability by using random forests and regression forests techniques*, Expert Systems with Applications, 29(2), (2005), 472–484.

[12] N. Levin, J. Zahavi, *Predictive modeling using segmentation*, Journal of Interactive Marketing, 15(2), (2001), 2–22.

[13] G. S. Linoff, M. J. Berry, *Data mining techniques: for marketing, sales, and customer relationship management*, John Wiley & Sons, 2011.

[14] Los Alamos National Laboratory, *Graphic Math*, Available online [2016/07/01], http://www.lanl.gov/discover/publications/1663/2013-july/graphic-math.php

[15] M. E. J. Newman, *The structure and function of complex networks*, SIAM review, 45(2), (2003), 167–256.

[16] K. Ng, H. Liu, *Customer retention via data mining*, Artificial Intelligence Review, 14(6), (2000), 569–590.

[17] E. W. Ngai, L. Xiu, D. C. Chau, *Application of data mining techniques in customer relationship management: A literature review and classification*, Expert Systems with Applications, 36(2), (2009), 2592–2602.

[18] C. Phua, V. Lee, K. Smith, R. Gayler, *A comprehensive survey of data mining-based fraud detection research.*, arXiv preprint arXiv:1009.6119, 2010.

[19] K. A. Smith, R. J. Willis, M. Brooks, *An analysis of customer retention and insurance claim patterns using data mining: A case study*, Journal of the Operational Research Society, 51(5), (2000), 532–541.

[20] WP - map, *Map of Slovakia*, Available online [2016/07/01], http://www.wpmap.org/map-of-slovakia/

The process approach in the education and the creation of the curriculum

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***Abstract*—The process approach is applicable to all organizations, regardless of type, size and products. A difference between university and industrial factories is also taken into consideration. The target of universities is to offer an education, which is realized effectively, verifiable - measurable and valuable. Process management principles are mainly oriented to meet the needs and expectations of customers. A customer, in the learning environment student is entitled to expect the best quality service; it is providing best quality education. The quality of the educational process is highly dependent on correct preparation of the process, content and methodological aspects on the one hand, formal, technical and other aspects on the other hand. The aim of this article is to contribute to improving the quality of the educational process through the application of process approach in creation of study plans.**

***Index Terms*—Education, Process Approach, Creating of Curriculum**

I. Introduction

Each organization should have its target and philosophy for which it was created. Management of the organization should contribute to reach this objective the most effective, in required time, in verifiable manner and at a reasonable means. Manufacturing organizations have the aim to earn money, focus on value producing processes. Processes that do not produce value are eliminate. The target of the universities is to provide special services education, that is realized efficiently and effectively.This can be achieved by focusing on the process.

II. The process approach in the education

For working organization is necessary to identify and manage its activities, processes for its customers. The process is any activity, or set of activities, that uses resources to transform inputs to outputs. The systematic identification and management of the processes employed within an organization and particularly the interaction between such processes is referred to ask the process approach.

The process approach is the one of the eight quality management principles of the international standards of quality management system ISO 9000 family [4]:

- 9000:2015 Quality management systems. Fundamentals and vocabulary.
- 9001:2015 Quality management systems. Requirements.
- 9004:2009 Managing for the sustained success of an organization. A quality management approach.
- IWA 2:2007 Quality management systems. Guidelines for the application of ISO 9001:2000 in education. (withdrawn)

The application of process approach is the same in production and services areas. There also should be into consideration difference between university and industrial factory.

- The university has suppliers from different types of secondary schools with mutually incomparable level of input knowledge. Number of students from vocational schools with graduation is increased and a number of students from grammar school is decreased.
- University is not an industrial factory; it is difficult to measure an added value.
- The process of education provides university teachers, who are owners of the process, but their basic qualification is not pedagogical.
- Educated students, are not products of education.
- The product of education is knowledge, independence and creative thinking.
- Product of education has several customer groups - consumers:
  - Students;
  - Potential employers;
  - Society as a whole.
- The aim of education is not profit, even though the financial costs it is important;
- Usually individuals who receive education are not the ones who make the payments.

An advantage of the process approach is the ongoing control that it provides over the linkage between the individual processes within the system of processes. By this will reach:

- Interaction and accordance of processes for achieving the aim;
- Growth of efficiency and effectiveness;
- Growth of confidence;
- Growth of productivity;
- Growth of responsibility;
- Transparently and achieving of continuous better and more awaited results.

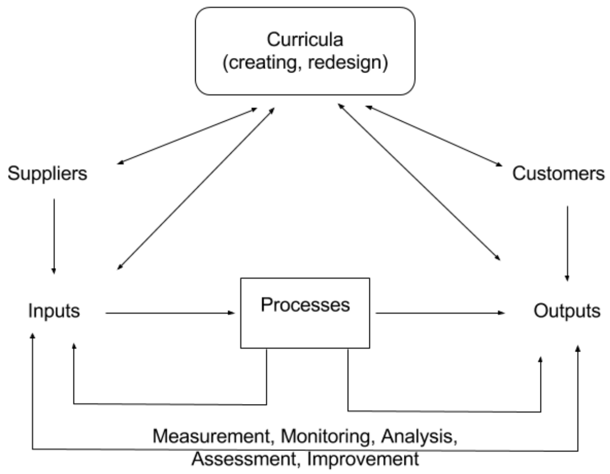

Fig. 1. Model of process approach in education [2]

As support of the process approach serves interaction between [3]:

- Inputs and suppliers of inputs;
- Owners, realizations of transformation;
- Outputs to customers of outputs;
- Mass and individual customers, external and internal customers;
- National and international legislation;
- Information and communication technologies;
- Technologies and techniques in comparable level.

The educational process how each specific process includes inputs and outputs. Process approach model in education is illustrated on Fig.1. All processes between input and output should be efficient - capable to achieve the required results and efficient - capable to provide a minimum difference between the achieved results with the required resources and original requirements. Education process is a group of processes:

- Education;
- Learning;
- consultation;
- Laboratory work;
- Research.

Suppliers in the educational process are [1], [5]:

- Applicants for a study;
- Applicants Parents;
- High schools;
- Ministry of Education;
- External teachers;
- Sponsors and consultants;
- Information funds;
- suppliers for operation.

It is important to realize, that University cant select supplier. The research shows that in recent years number of students coming from high schools is decreasing and the number of students coming from vocational schools is increasing; Its results in the general decline of student initial knowledge level.

The outputs in education process represent the values of obtained by graduates. Students involved in the education process with specific skills, knowledge and levels of knowledge. Students leave from the education process with added value. This added value is:

- Competence;
- Application skills;
- General knowledge - technical, economic, legal, management;
- Competitiveness in the labor market
- Recognition of qualification;
- Mature person.

The outputs of the educational process are perceived by customers. The customer entities are:

- Students;
- Parents of students;
- Future employers;
- Public;
- State and registry organizations;
- Managers and employees.

III. The application of the process approach in creation of curriculum

Quality of the educational process strongly depends on the preparation of the process, on understanding of aspects of contents, methods, forms, techniques and other sides of the process.

The customer is taken like the most important element within creating of educational programmers. In this case the customer is a student. Customer has a claim to get the best service from a teacher. It means the most quality education.

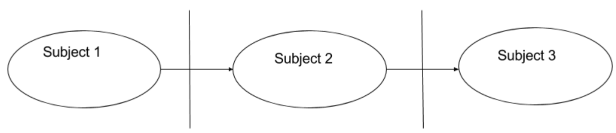

Fig. 2. Invisible wall between subjects

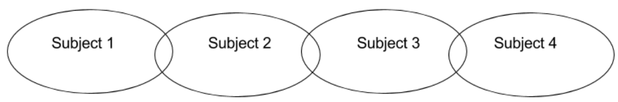

Fig. 3. Overlapping of subject contents

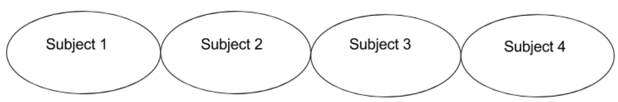

Fig. 4. Optimal process of creation of curricula

In Study plans are several serious problems at present. This problems are illustrated on Figures 2-4.

- *Invisible wall* - each subject is an independent unit, and continuity is losing; Students are receiving "isolated" knowledge and skills. Knowledge is accumulated without systematic continuity.(Fig.2).
- *Overlapping of subject contents.* The content of one subject overlaps with the content of another subject. Students often prefer it. Poor communication between teachers causes loss of time 9but repeating is mother of wisdom). This problem is often in today's situation. (Fig.3)
- *optimal process* is the process, which is continual in the content and in the time, continual in subjects in semesters. Such process is balanced, without "narrow places", with the synergistic effect.(Fig.4)

IV. Conclusion

One of the criteria of nation‘s education is the percentage expressing number of people, who have achieved university education. MAin aim is not only to get diploma, but to obtain the highest quality of education.Quality is necessary, but not sufficient for all activities. Therefore, preparation of the educational process is important.By creating of curriculum is necessary to minimize repetition, overlapping of subjects content and to remove invisible wall among subjects.

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References

[1] G. Bogdanovská, *Quality assurance of the education process in the extraction and processing of raw materials*, Dissertation thesis, Faculty of Mining, Ecology, Process Control and Geotechnology, Technical Univerzity in Kosice, Kosice, 2009.

[2] G. Bogdanovská, L. Floreková, M. Čarnogurská, *The aplication of process approach for desing of study planes*, Acta Montanistica Slovaca, 4, (2006), 293–297.

[3] R. W. Peach, B. Peach, D. S. Ritter, *Memory Jogger*, 9000/2000, GOAL QPC, SSQ, 2002.

[4] ISO 9001 Quality Management Systems. Requirements.

[5] J. Zeleny, *Fakulta a kvalita*, TU Zvolen, Zvolen, 1998.

Geographic information systems in university education

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***Abstract*—Geographic information systems (GIS) represent a powerful tool that allows collecting, process, analyze, understand and interpret wide range of data from very different areas of human life. It can be conceived also as a tool to resolve certain problems, a tool for research and modeling, as well as a set of certain thematic layers required for further processing. Here we want to point out at the need of education in GIS environment, emphasize the importance of involvement of GIS in the educational process as well as to motivate the students from different fields of study to versatile usage of GIS.**

***Index Terms*—Geographic Information System, Usage of GIS, Education process**

I. Introduction

Geographic or Geospatial information systems, shortly GIS, are computer systems that enable users to capture, store, analyze and manage spatially referenced data. In the traditional meaning, GIS's have transformed the geographic data, relationships and patterns into quickly and easily understandable map. Nowadays the powerful tool that is behind GIS is comprehend as the one that allow to capture, manage, view, understand, query, process, analyze, interpret, map, model, display and visualize our world for an increasingly large range of users in ways that reveal relationships, patterns, and trends in the data visualization.

II. Teaching GIS at University of Prešov in Prešov

The GIS are today needed not only in research institutions for environmental scientists and land use planners. Widespread communities from health organizations through businesses up to government agencies at all levels found applications of GIS, following the prefigurement of technical universities. Since the year 2001 GIS are being taught also at the Prešov University in Prešov. Department of Geography and Applied Geoinformatics garrantee 5 study programs in internal form (of study): 2 bachelor study programs - Geography Teaching in the field of study 1.1.1. - Academic Subjects Teaching and Geography and Geography and Applied Geoinformatics in the field of study 4.1.35. - Geography; 2 master study programs - Geography Teaching in the field of study 1.1.1. - Academic Subjects Teaching and Geography and Applied Geoinformatics in the field of study 4.1.35. - Geography; 1 PhD. study program (internal and external form) is focussed on Regional Geography and Regional Development in the field of study 4.1.38. - Regional Geography.

In the 1st semester of the bachelor study program Geography and Applied Geoinformatics, students learn to work with an existing database and try to construct simple cartographic outputs. In the second semester they learn how to analyze the data. Afterwards students have to choose from three modules: Tourism and Territorial Marketing, Regional Development and Regional Policy, and Landscape Plannning and Landscape Management. Despite the fact that a substantial part of the GIS will be taught in the subjects named Applied Geoinformatics 1-4, the students are comming with touch with GIS in almost all subjects taught during the entire duration of study. The amount of teaching hours depends on the selected module and particulary from the content of the subject. Starting from the school year 2015/2016 the Department of Geography and Applied Geoinformatics access to use the Open Source GIS technologies due to low operating costs.

Students of the second bachelor study programm learn to work with GIS via school-subjects Creation of Maps by Computer 1 and 2. The first one is focussed on working with existing data. Selective subject Creation of Maps by Computer 2 continues with creation of the own database, georeferencing and vectorization.

III. Usage of GIS

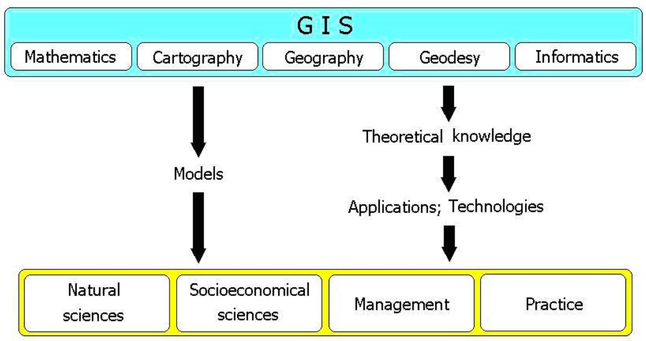

Fig. 1. Characteristics of GIS (Source: based on [14])

As the usage of visual presentation of spatial relationships in

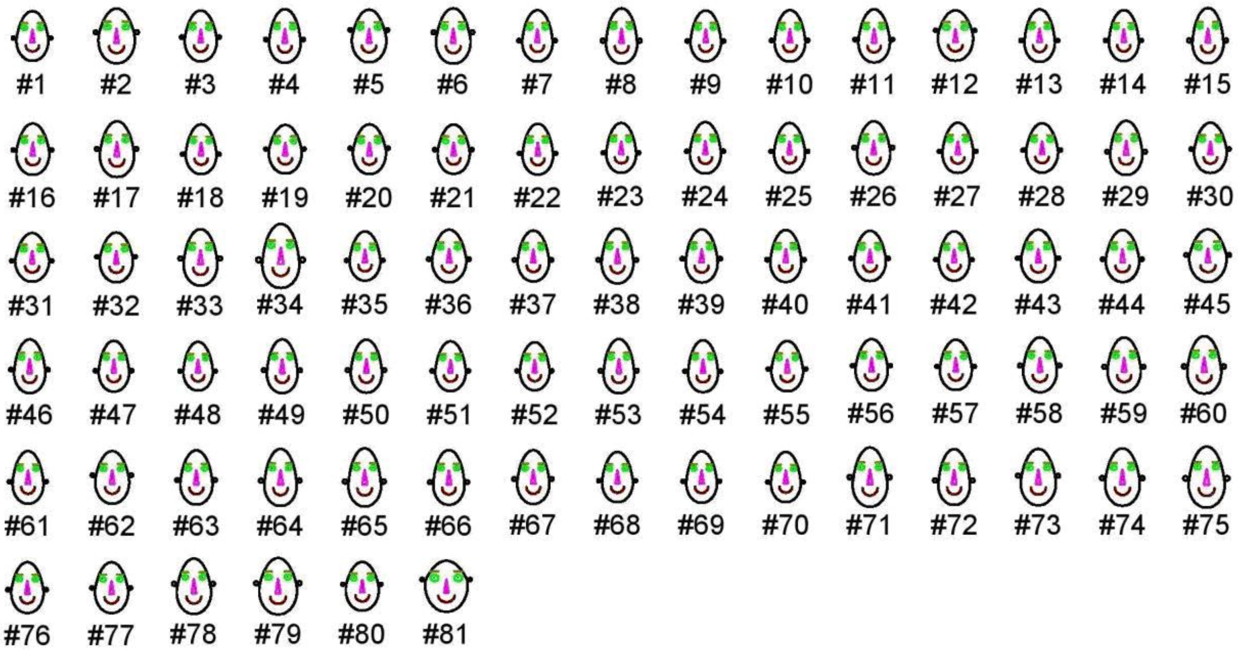

Fig. 2. Chernoff faces (icon-graph) of EU universities similarities assessment (Source: Authors research)
Width of the face = number of students, height of ears = number of pedagogues, height of the face = number of faculties at the university

the last decades got far beyond the borders of geography, we consider this being a good step forwards. In every situation where one has to process, manage, analyze and interpret statistical data related to some place on the Earth (or even Universe), it is possible to visualize them via tools of GIS. Therefore, it is worth to get some knowledge about GIS already during the study at university, no matter whether the aim is to become geographer, scientist, teacher, economist or other kind of engineer. The geographical information systems were used worldwide in very different areas, for example in crime mapping [11], [13], road networking [4], waste management [12], climatology and hydrology [8], [10], environmental applications [7] and elsewhere.

Our aim is to show our students widespread extent of GIS not only in geodesy and geography, but also in other scientific disciplines. GIS stands as a powerful tool also in social sciences, where the data processing is required for mapping and spatial modeling [2], [9]. For example in one our research we agglutinated the geographical information systems with geography, sociology, mathematics and mathematical statistics. We worked on the analysis of the similarity's rate of 81 universities and other schools of university's type in European Union (EU). From each state of the EU we have chosen three largest universities (when there were three universities in the considered state). The main criterion of choice was the number of students of the university. For each of these universities we found and add to our database information about the name of the university, the state and city where it is located, number of students, pedagogues and faculties, as well as the possibilities to study mathematics, geodesy, geography and geoinformatics there and at which faculty/faculties. We aim our attention on these subjects as there is a close relationship between them. Sudolská describes this relationship via Figure 1 (see [14]).

Having the statistical database, we wanted to process, analyse and compare the elements of this multidimensional statistical set. There are several ways how to do it - see [1], [3]. We processed the data using hierarchical clustering methods based on the simple join of several clusters. Similarly, we used methods of explorative analysis presented by profile diagrams in order to compare chosen 81 universities (see [5], [6] and [15]). As a result we obtained a large amount of maps, cartograms, cartodiagrams, dendrographs and diverse icon-graphs based on which we were able to compare the universities - see for example the icon-graph of EU universities similarities assessment at Figure 2 and a comparation of the number of students and number of teachers at selected universities of the European Union at Figure 3.

We showed how the space analysis created in the geographical information system environment can be joined with the methods of mathematical statistics. This is important both from the interdisciplinary and the statistical-environment analysis point of view, as it is proof of the fact that geographical information systems together with multidimensional statistical methods can be used also at interdisciplinary level.

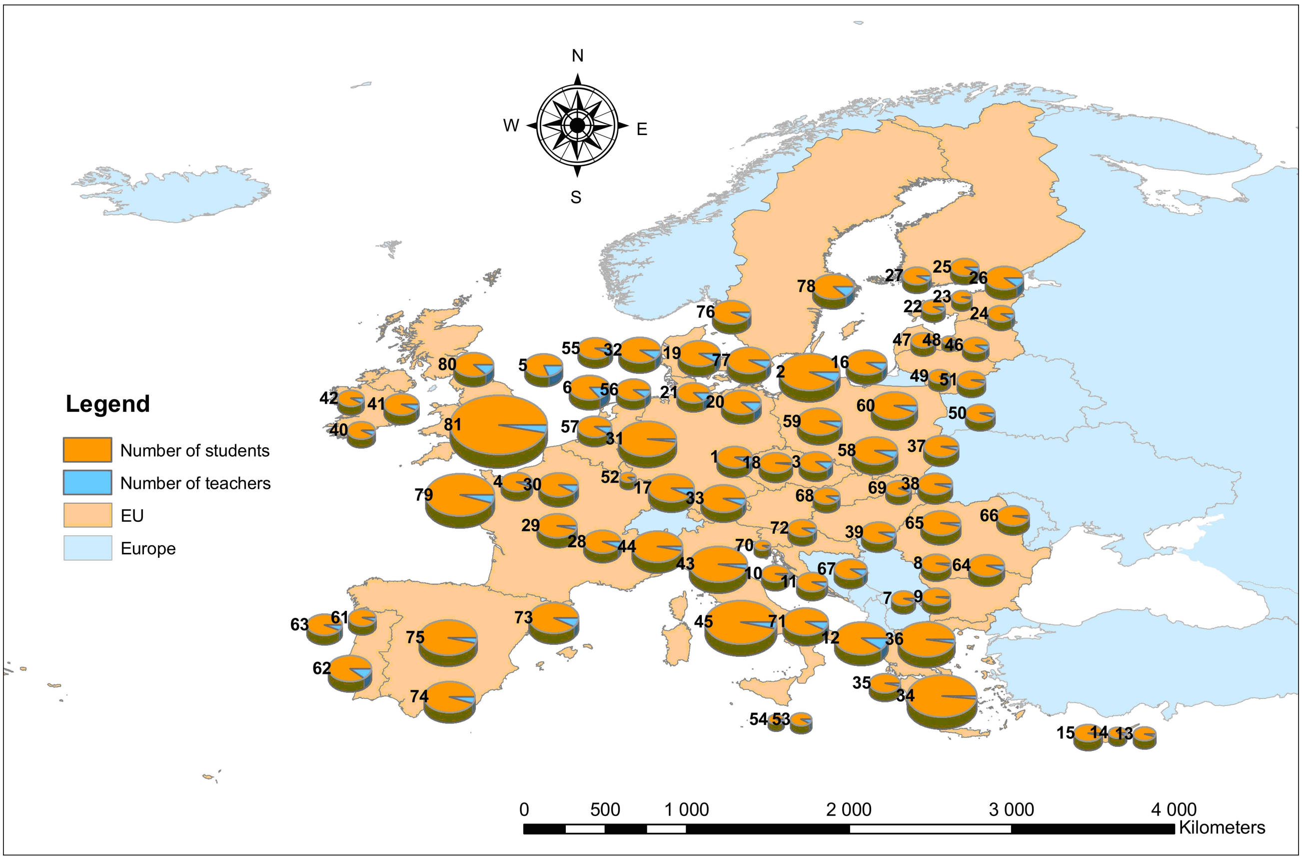

Fig. 3. Rate of the number of students and number of teachers at selected universities of the European Union (Source: [15])

This example shows that already students of the first degree of university education are able effectively use the geographical information systems and visualize and interpret the data processed parallely via GIS and multidimensional statistical methods using software Statistica (see [6]).
But this is only a single example how the geographical information systems taught during the education process can be used in order to produce fruitful output - comparative study. (Another example one can find in e. g. [5].) There are much more ways how to use the GIS in the educational process and afterwards in praxis.
The maps are extremely useful as a method of transmitting complicated information to communities. Usage in the geography, geodesy, climatology and hydrology is clear. In the biology we can use it for example in order to map the geographical location of species, while maps of zones with the high risk of epidemics are of use in health management and medicine. In history one can via GIS represent e.g. attempts, successes and failures of some conqueror. A wide area of utilization is also in mathematics and geometry where one can measure distances in the map based on spherical geometry. In the physics one can study the propagation of light, air-borne spread of a sound and other.

IV. Conclusion

In the present paper we pointed out to the wide spread areas where one can use GIS. We underlined that GIS represent a powerfull tool not only for people dealing with geography and cartography, but also for those, who are dealing with natural sciences, socioekonomical sciences and other scientific disciplines.
We are happy that we can motivate our university students to study the geographical information systems and related subjects. We hope, that the outputs of it would be even more rich after the change in the form of a new study block like system, what academic year 2015/2016 brought to the Prešov University in Prešov.

We agree also with Sudolská [14], that even if it is beyond the State Education Program in Slovak Republic, it would be worth to teach the basics of GIS also at secondary grammar schools and high schools at least at voluntary base at some seminars, hobby groups or via project-learning, and that it is doable to find a space for GIS in educational process.
Afterwards, it requires only computer usage, well defined own research problem (the formulation of a task), some knowledge about interpretation of obtained results, and valuable report, map or analysis will be spawned.

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REFERENCES

[1] D. Asimov: *The grand tour: A tool for viewing multidimensional data*, SIAM Journal on Scientific and Statistical Computing, 6(1), (1985), 128–143.

[2] M. Boltižiar, M. Vojtek: *Geografické informačné systémy pre geografov II (Geographical information systems for geographs II)*, Nitra, Fakulta prírodných vied Univerzity Konštantína Filozofa v Nitre, 2009.

[3] B. M. Collins: *Data visualization - has it all been seen before?*, In: R. A. Earnshaw, D. Watson, (Eds.), *Animation and scientific visualization: Tools and applications*, Academic Press, New York, (1993) 3–28.

[4] M. Ebrahim, A.-E. Ihab: *Planning road networks in new cities using GIS: The case of new sohag*, Egypt, International Journal of Geoinformatics, 7(3), (2011).

[5] E. Fecková Škrabuľáková, M. Ivanová, E. Michaeli: *Usage of clustering methods in mathematics, geoinformatics and related fields of university study*, In: Proceedings of the 17th International Carpatian Control Conference (ICCC), May 29 - June 1, 2016, Grandhotel Praha, High Tatras, Slovak Republic, (2016), 723–728.

[6] A Gallovičová, *Využitie GIS v príbuzných odboroch geodézie (Utulization of GIS in the subject fields related to geodesy)*, F BERG TUKE, Bachelor Thesis - unpublished manuscript, 2014.

[7] J. C. Hinton, *GIS and remote sensing integration for environmental applications*, International Journal of Geographical Information Systems, 10(7), (1996), 877–890.

[8] I. Hrnjak, T. Lukić, M. B. Gavrilov, S. B. Marković, M. Unkašević, I. Tošić, *Aridity in Vojvodina, Serbia, Theoretical and applied climatology*, 115(1-2), (2014), 323–332.

[9] N. S.-N. Lam, *Spatial interpolation methods: A review*, The American Cartographer, 10(2), (1983), 129–150.

[10] D. R. Maidment, *GIS and hydrologic modeling*, In: M. F. Goodchild, B. O. Parks, L. T. Steyaert, (Eds.), *Environmental Modeling with GIS*, Oxford University Press, New York, (1993), 147–167.

[11] M. D. Maltz, A. C. Gordon, W. Friedman, *Mapping crime in its community setting: Event geography analysis*, (Internet ed.), Springer-Verlag, New York, 2000.

[12] S. M. Ramasamy, C. J. Kumanan, K. Palanivel, *GIS-based solutions for waste disposals*, GISdevelopment.net, India, 2002.

[13] Ch. Spencer, J. Ratcliffe, *GIS and crime mapping*, John Wiley & Sons, 2005.

[14] M. Sudolská, *Geografické informačné systémy vo vyučovaní, (Geographical Information Systems in Education)*, 2. celoštátna konferencia INFOVEK 2001, 17.-20. X. 2001, Stará Turá (online), 2001.

[15] E. Škrabuľáková, M. Ivanová, A. Gallovičová, *An utilization of the GIS and multidimensional statistical methods on the chosen EU universities: Similarities assessment*, Geographica Pannonica, September 2014, (2014), 62–68.

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